A Remarkable Formula, and Its Applications

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The purpose of this document is to tie together a few threads from this course. Trigonometric functions, exponential functions, complex numbers and polynomials are all intimately related to one another. Let me take you on a quick tour.

1 The Keystone

There is the remarkable formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Why is this true? I cannot answer that right now. If you continue studying math, you will find out in calculus II.

But this formula really is neat. Check this out:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

That's right. $e^{i\pi} + 1 = 0$. That's unexpected and maybe cool, but it is probably not so useful. Are there applications of this formula that are useful?

2 The Master Trigonometric Identity

We will be embarking on a study of trigonometric identities. A couple of things to remember about such identities: First, there are infinitely many of them. Second, all of the most important ones come straight from the keystone above.

Example: Calculate $e^{i\theta}e^{i(-\theta)}$.

Solution: On the one hand we expect that $e^{i\theta}e^{i(-\theta)} = 1$ by addition of exponents. Keep this in mind. On the other hand:

$$e^{i\theta}e^{i(-\theta)} = (\cos\theta + i\sin\theta)(\cos(-\theta) + i\sin(-\theta)) =$$
$$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta.$$

Therefore,

$$\cos^2\theta + \sin^2\theta = 1.$$

How about another example. Let's try

Example: Calculate $e^{i\theta}e^{i\phi}$.

Solution On the one hand we expect that

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi).$$

On the other hand:

$$e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) =$$

$$(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi).$$

Thus,

$$\cos(\theta+\phi) + i\sin(\theta+\phi) = (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi).$$

Now, two complex numbers are equal if and only if their real and imaginary parts are equal. Therefore we pick up two identities:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$

and

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

All of the other formulas given in sections 7.3-7.6 of the textbook can be found by similar arguments.

So, this is the master trigonometric identity. This is cool and maybe a little useful (but do we really want to re-derive these identities all the time?). Nonetheless, perhaps we were hoping for a bigger payoff. There is some payoff coming. First we need a little detour.

3 Visualizing the Complex Numbers

As we have seen, we can invent a number i such that $i^2 = -1$. The question arises: where does this strange number i appear on the number line? Perhaps we would feel better about i if we could "locate" it.

The answer is that i is not on the number line at all. Instead, i lies one unit above the number line. More specifically, we create the "number plane" (more usually called the complex plane), where the x-axis measures the real part of the number and the y-axis measures the imaginary part. For example, we can plot $2 - 3i$ on the number plane as follows:

This provides us with a visualization of complex numbers analogous to rectangular coordinates. To wit: we can convert a point on the plane to a complex number, or vice versa. For example, the number (-1,-5) could be interpreted as the complex number $-1-5i$.

In this formulation, addition (and subtraction) has an intuitive geometric meaning. Given 2 complex numbers $z_1 = a+bi$ and $z_2 = c+di$, plot the two points and draw arrows to them. Then make the parallelogram with corners at 0, $a+bi$ and $c+di$. The fourth corner is $z_1+z_2 = (a+c)+(b+d)i$. The two figures below show this process for the numbers $-1-2i$ and $5+i$. Notice that the sum of these two is $4-i$.

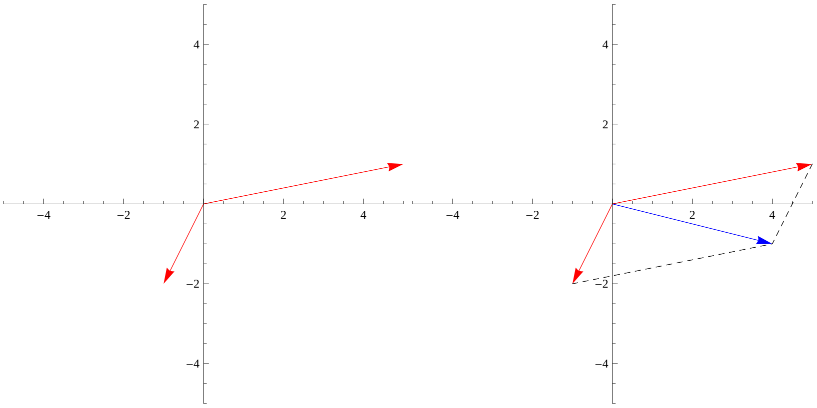

Figure 1: Visualizing the Addition of Two Complex Numbers

What about multiplication? Multiplication isn't so easy if we're thinking in rectangular terms, but it does have a nice geometric interpretation if we reorient our thinking...

4 Polar Coordinates

If we have a point in the plane, we are used to describing it by telling the horizontal distance from the origin (the x-coordinate) followed be the vertical distance from the origin (the y-coordinate). There is another way: instead, we could measure the total distance from the point to the origin. Let's call this r. We then have

$$r = \sqrt{x^2+y^2}.$$

If we just know r, we can't identify the point. The other coordinate we need is θ, which measures the angle the point makes with respect to the x-axis. (If this seems strange, think about how pilots talk: "bogey 500 feet away at 3:00." Here, $r = 500$ ft and 3:00 identifies the direction.) This system is called polar coordinates.

Then, we can convert from polar to rectangular by the formulas

$$x = r\cos\theta,$$

$$y = r\sin\theta.$$

That is, if we know r and θ, the point we're talking about in rectangular coordinates is $(r\cos\theta, r\sin\theta)$. As a complex number, this is

$$r\cos\theta + ir\sin\theta = re^{i\theta}.$$

It will also be very useful to convert the other way. If we know x and y, can we find r and θ? We can. The formulas are

$$r = \sqrt{x^2+y^2},$$

$$\theta = \begin{cases} \arctan\frac{y}{x} & x > 0 \\ \pi + \arctan\frac{y}{x} & x < 0 \end{cases}.$$

If $x = 0$, then θ is either $\frac{\pi}{2}$ (if $y > 0$) or $\frac{3\pi}{2}$ (if $y < 0$).

5 Multiplication

Now, about complex multiplication: Let $z_1 = r_1 e^{i\theta_1}$, and $z_2 = r_2 e^{i\theta_2}$. Then

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)}.$$

In other words: When we multiply two complex numbers together, the new length is the product of the old lengths and the new angle is the sum of the old angles.

Example: Calculate $(-1+\sqrt{3}i)(\frac{1}{2}-\frac{1}{2}i)$.

Solution: Let $z_1 = -1+\sqrt{3}i$ and $z_2 = \frac{1}{2}-\frac{1}{2}i$. We'll start by converting these to polar form.

$$z_1 = 2e^{\frac{2\pi}{3}i}$$

$$z_2 = \frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}i}$$

Now, multiply.

$$z_1 z_2 = 2e^{\frac{2\pi}{3}i}\frac{\sqrt{2}}{2}e^{-\frac{\pi}{4}i} =$$

$$\sqrt{2}e^{(\frac{2\pi}{3}-\frac{\pi}{4})i} = \sqrt{2}e^{\frac{5\pi}{12}i}$$

In other words, the result has a length of $r = \sqrt{2}$ and makes an angle of $\theta = \frac{5\pi}{12}$. It would usually be good to convert this to rectangular coordinates (that is, make it of the form $a + bi$), but taking $\sin\frac{5\pi}{12}$ and $\cos\frac{5\pi}{12}$ is not so easy (it requires trigonometric identities like perhaps the half angle formulas). Nonetheless, look at the geometry of this situation, particularly the angles:

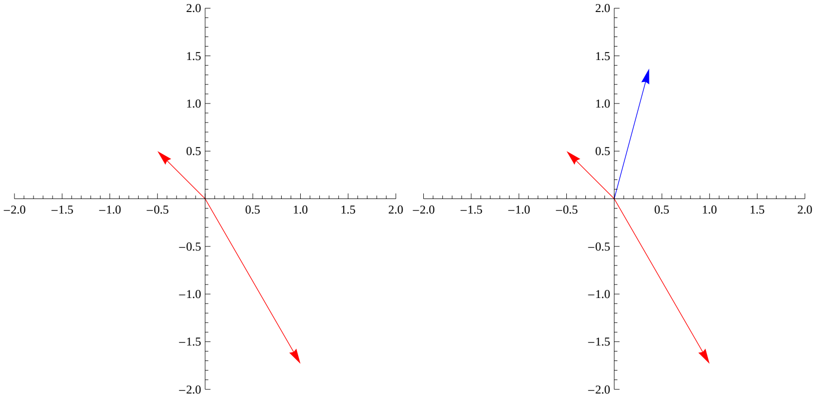

Figure 2: Visualizing the Multiplication of Two Complex Numbers

As a consequence of this viewpoint, notice that if we have a complex number in polar form $z = re^{i\theta}$, then we have

$$z^2 = r^2 e^{i(2\theta)},$$

$$z^3 = r^3 e^{i(3\theta)},$$

and generally,

$$z^n = r^n e^{i(n\theta)}.$$

6 Roots of 1

The next place where this wonderful formula comes in handy is that it will help us to find all of the n-th roots of any complex number. We will start by finding all of the n-th roots of 1.

Finding the n-th roots of 1 means solving the equation

$$x^n = 1 \text{ or, rewriting: } x^n - 1 = 0.$$

We expect there to be n roots since this is a degree n polynomial equation. How shall we get at them, though? Let's try a few easy examples.

$n = 1$: There is only 1 1st root of 1, namely 1. This has length 1 and can be written as $1 = 1e^{0i}$; that is, $r = 1$ and $\theta = 0$.

$n = 2$: There are two square roots of 1: 1 and -1. These can be written as

$$1 = 1e^{0i}$$

$$-1 = 1e^{\pi i}.$$

$n = 3$**:** We want to solve $x^3 - 1 = 0$, and we know that 1 is one of the solutions. In fact,

$$x^3 - 1 = (x-1)(x^2 + x + 1).$$

Using the quadratic formula on the second factor yields the roots $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$. These roots can be written as

$$1 = 1e^{0i}$$

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}i = 1e^{\frac{2\pi i}{3}}$$

$$-\frac{1}{2} - \frac{\sqrt{3}}{2}i = 1e^{\frac{4\pi i}{3}}.$$

$n = 4$**:** We want to solve $x^4 - 1 = 0$. We can factor as follows:

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x+1)(x-1)(x+i)(x-i).$$

The four roots are therefore 1, -1, i, and $-i$. These roots can be written as

$$1 = 1e^{0i}$$

$$i = 1e^{\frac{\pi}{2}i}$$

$$-1 = 1e^{\pi i}$$

$$-i = 1e^{\frac{3\pi}{2}i}.$$

$n = 5$**:** Too hard for us at the moment...

Now, let's plot these complex numbers to find a pattern.

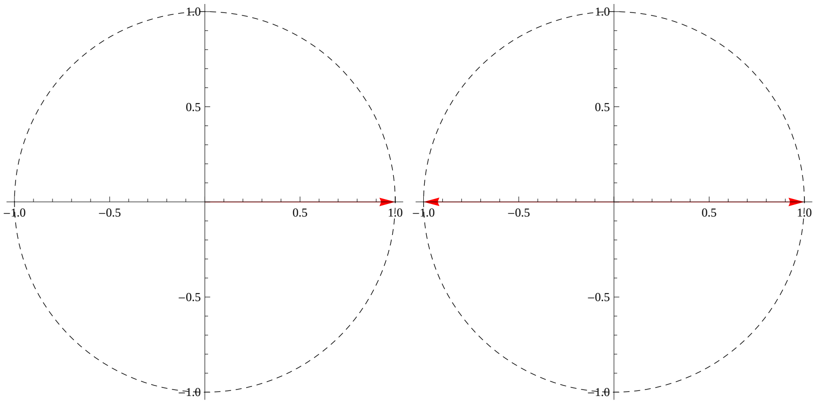

Figure 3: Left: 1st root of 1. Right: 2nd roots of 1

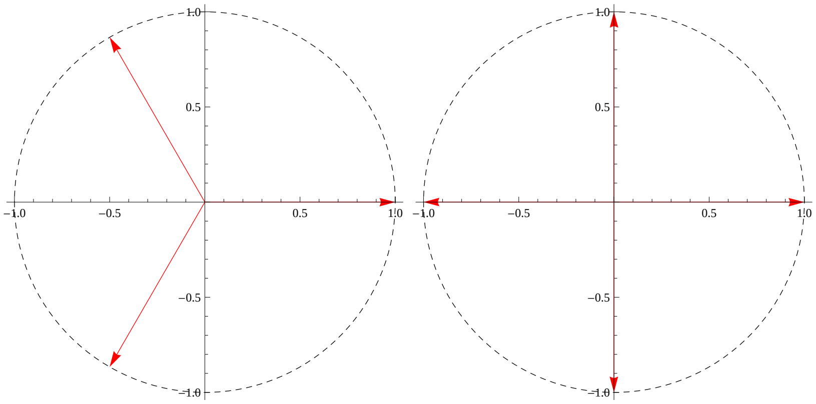

Figure 4: Left: 3rd roots of 1. Right: 4th roots of 1

Notice that the n roots are equally spaced around the unit circle in each case. To be specific, we have the following principle: The nth roots of 1 are given by

$$1e^{\frac{2\pi ik}{n}} \text{ for } k = 0, 1, 2, \cdots, n-1.$$

For $n = 5$, we get the following roots of 1:

$$e^{\frac{2\pi i\cdot 0}{5}} = 1$$

$$e^{\frac{2\pi i\cdot 1}{5}} = e^{\frac{2\pi i}{5}}$$

$$e^{\frac{2\pi i\cdot 2}{5}} = e^{\frac{4\pi i}{5}}$$

$$e^{\frac{2\pi i\cdot 3}{5}} = e^{\frac{6\pi i}{5}}$$

$$e^{\frac{2\pi i\cdot 4}{5}} = e^{\frac{8\pi i}{5}}$$

7 nth Roots of any Complex Number

Let's assume we have a complex number in polar form $z = re^{i\theta}$. It is actually very easy to get one nth root:

$$\sqrt[n]{z} = \sqrt[n]{re^{i\theta}} = (re^{i\theta})^{\frac{1}{n}} = r^{\frac{1}{n}}e^{\frac{i\theta}{n}}.$$

What about the other roots? Those can be found by multiplying the one root by all nth roots of 1.

Example: Find all six of the sixth roots of $64i$.

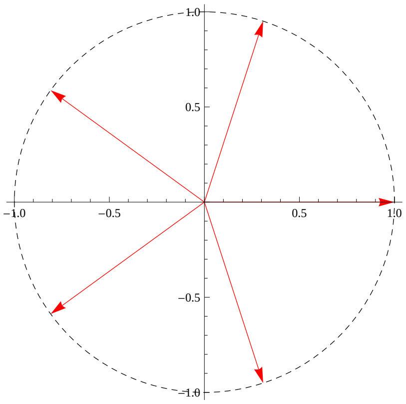

Figure 5: 5th roots of 1

Solution: Start by converting to polar form:

$$64i = 64e^{\frac{\pi}{2}i}.$$

Now find a sixth root of i:

$$\sqrt[6]{64i} = \sqrt[6]{64e^{\frac{\pi}{2}i}} = 2e^{\frac{\pi}{12}i}.$$

Next, find all of the sixth roots of 1:

$$\begin{gathered} e^{\frac{2\pi i\cdot 0}{6}} = 1 \\ e^{\frac{2\pi i\cdot 1}{6}} = e^{\frac{\pi}{3}i} \\ e^{\frac{2\pi i\cdot 2}{6}} = e^{\frac{2\pi}{3}i} \\ e^{\frac{2\pi i\cdot 3}{6}} = e^{\pi i} = -1 \\ e^{\frac{2\pi i\cdot 4}{6}} = e^{\frac{4\pi}{3}i} \\ e^{\frac{2\pi i\cdot 5}{6}} = e^{\frac{5\pi}{3}i} \end{gathered}$$

Finally, take the sixth root of i and multiply it be each of the sixth roots of 1:

$$\begin{gathered} 2e^{\frac{\pi}{12}i}\cdot 1 = 2e^{\frac{\pi}{12}i} \\ 2e^{\frac{\pi}{12}i}\cdot e^{\frac{\pi}{3}i} = 2e^{\frac{5\pi}{12}i} \\ 2e^{\frac{\pi}{12}i}\cdot e^{\frac{2\pi}{3}i} = 2e^{\frac{9\pi}{12}i} = 2e^{\frac{3\pi}{4}i} \\ 2e^{\frac{\pi}{12}i}\cdot e^{\pi i} = 2e^{\frac{13\pi}{12}i} \\ 2e^{\frac{\pi}{12}i}\cdot e^{\frac{4\pi}{3}i} = 2e^{\frac{17\pi}{12}i} \\ 2e^{\frac{\pi}{12}i}\cdot e^{\frac{5\pi}{3}i} = 2e^{\frac{21\pi}{12}i} = 2e^{\frac{7\pi}{4}i} \end{gathered}$$

A motivated student might convert the third and sixth to rectangular cordintes. Once more, let's look at the geometry of the situation:

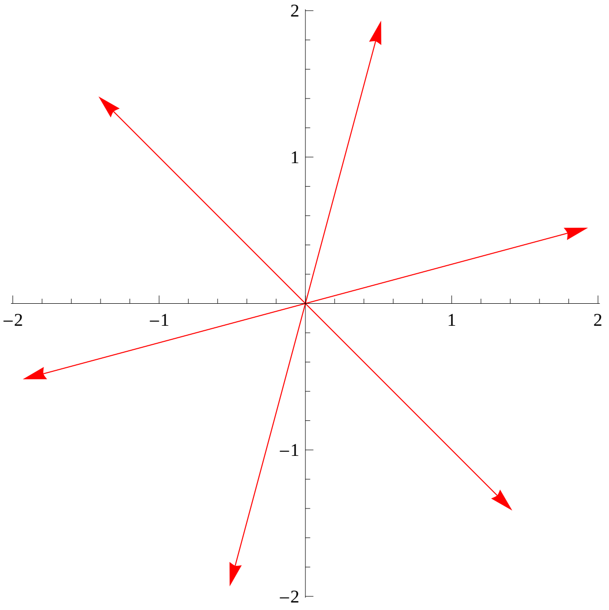

Figure 6: 6th roots of i