Review of the Arithmetic of Fractions

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1 Prime Numbers

A prime is a whole number which is only divisible by one and itself. The first few prime numbers are $2, 3, 5, 7, 11, 13, 17, 23, 29, 31$.

The easiest way to tell if a number is prime is to take the square root of that number and attempt to divide each prime less than the square root into the number. If none of these primes produce an even division, then the original number is prime.

For example, let us say that we are trying to determine whether the number 109 is prime. The square root of 109 is between 10 and 11. Therefore, we only need to try to divide the primes 2, 3, 5, and 7 into 109 (11 is the next prime, and is bigger than the square root of 109). We find that none of the divisions $109 \div 2$, $109 \div 3$, $109 \div 5$ and $109 \div 7$ produce an whole number, which indicates that none of these primes divide 109 evenly. Therefore, 109 is indeed a prime number.

2 Prime Factorizations

We have the impressive sounding Fundamental Theorem of Arithmetic: every positive integer has a unique factorization into prime numbers.

How do we get this prime factorization? Let us consider the number 80. We can immediately tell that 80 is not prime, and so we must factor it. One way to do this is to realize that $80 = 2 * 40$. We know that 2 is prime and 40 is not, and so 40 needs further factoring. We see that 40 is divisible by 2, and we can write $40 = 2 * 20$. This means that $80 = 2 * 40 = 2 * 2 * 20$. Again, the 2 is prime, but the 20 is not, and so we continue factoring: $20 = 5 * 4 = 5 * 2 * 2$. Notice here that all the factors (2 and 5) are prime. Therefore, $80 = 2 * 2 * 2 * 2 * 5$ is the prime factorization.

What if we want the prime factorization of 13, which is already prime? Answer: the prime factorization of 13 is simply 13. We just can't factor it.

Special note: The number 1 has an empty prime factorization because it isn't prime and can't be factored.

3 Least Common Multiples

Let's say that we have two numbers and we want to find another number which is a multiple of both of them. For concreteness, assume we begin with the numbers $80 = 2 * 2 * 2 * 2 * 5$ and $210 = 2 * 3 * 5 * 7$. An easy way to find a common multiple is to simply multiply the two numbers together: $80 * 210 = 16800$. But 16800 is fairly large, and perhaps we can find a smaller number that's easier to work with. This is why we look for the least common multiple (LCM). The method of finding the LCM of two numbers is as follows:

1. Find the prime factorizations,
2. For each prime look at how many factors of that prime appear in each prime factorization and keep the larger number of factors,
3. Multiply it all together.

Let's do the example to illustrate this process. Notice that the prime factorization of 80 has four factors of 2, while the prime factorization of 210 has only one factor of 2. We keep the larger number of factors, so our least common multiple should have four factors of 2. Continuing, 80 has no factors of 3 while 210 has one factor of 3. Our LCM should therefore have one factor of 3. Cutting to the chase, we will also get one factor of 5 and one factor of 7. Putting it all together, the LCM of 80 and 210 is then $2 * 2 * 2 * 2 * 3 * 5 * 7 = 1680$.

It will also be of interest to figure out what number we had to multiply by to get to the LCM. Notice that in order to turn the prime factorization for 80 into the prime factorization of the LCM, we need an extra factor of 3 and an extra factor of 7. Therefore, $1680 = 80 * 3 * 7 = 80 * 21$. Similarly, to get from 210 to 1680 we need three extra factors of 2. Therefore, $1680 = 210 * 2 * 2 * 2 = 210 * 8$.

We had to multiply 210 by 8 to get to the LCM, and we had to multiply 80 by 21 to get to the LCM.

4 Greatest Common Divisors

The greatest common divisor (GCD) of two numbers is the largest number that divides evenly into both numbers. The method for finding the GCD is similar to the method for finding the LCM:

1. Find the prime factorizations,
2. For each prime look at how many factors of that prime appear in each prime factorization and keep the smaller number of factors,
3. Multiply it all together.

For an example, we again turn to the numbers 80 and 210. 80 has four factors of 2, while 210 has one factor of 2. We keep the smaller number of factors for the GCD, so our GCD will have one factor of 2. Continuing, 80 has no factors of 3, while 210 has one factor of 3. Therefore our GCD will have no factors of 3. Completing the process, our GCD will have one factor of 5 and no factors of 7. Putting it all together, our GCD becomes $2 * 5 = 10$.

What is the GCD of 3 and 5? We apparently get a GCD with no prime factorization. This means the GCD is 1.

5 Simplifying Fractions

The golden rule of fractions is this: whatever you multiply or divide the bottom of the fraction by, multiply or divide the same thing on top. And vice versa. Thus we can write $\frac{15}{10} = \frac{3}{2}$ if we divide the top and bottom by 5. We can also write $\frac{15}{10} = \frac{30}{20}$ if we multiply and top and bottom by 2.

We will prefer our fractions to be in lowest terms, so that the numerator and denominator are as small as possible. We accomplish this by finding the GCD of the numerator and denominator, and then dividing both of them by the GCD. For instance, to reduce the fraction $\frac{90}{1350}$ into lowest terms, note that 90 has a prime factorization of $2 * 3 * 3 * 5$ and 1350 has a prime factorization of $2 * 3 * 3 * 3 * 5 * 5$. The GCD of 90 and 1350 is therefore $2 * 3 * 3 * 5 = 90$, and so we will divide the numerator and denominator by 90. This will leave us a numerator of 1 and a denominator of 15.

$$\frac{90}{1350} = \frac{1}{15}.$$

If the GCD of the numerator and denominator happens to be 1, such as in the fraction $\frac{5}{2}$, it means that the fraction is already in lowest terms.

6 Adding and Subtracting Fractions

We might describe adding and subtracting fractions as a four step process.

1. Find the LCM of the two denominators,
2. Convert both fractions so that they have the LCM as their denominator,
3. Add or subtract the numerators while keeping the denominator the same,
4. Reduce to lowest terms.

For example, let us compute $\frac{7}{80} + \frac{5}{210}$:

1. The denominators are 80 and 210, and the LCM of 80 and 210 is 1680.
2. In order to convert 80 to 1680, we must multiply by 21. Recalling the golden rule, that means we also have to multiply the numerator by 21. Thus $\frac{7}{80} = \frac{147}{1680}$. To convert 210 to 1680, we must multiply by 8. Therefore $\frac{5}{210} = \frac{40}{1680}$.

3. Adding the new numerators produces $147 + 40 = 187$. Thus we get $\frac{187}{1680}$.
4. Finally, we divide the GCD out. The denominator 1680 has a prime factorization of $2 * 2 * 2 * 2 * 3 * 5 * 7$, while the numerator 187 has a prime factorization of $11 * 17$. Notice that our GCD is 1, so our answer is already in lowest terms. The answer is $\frac{187}{1680}$.

Summarizing the calculation, we might write

$$\frac{7}{80} + \frac{5}{210} = \frac{147}{1680} + \frac{40}{1680} = \frac{187}{1680}.$$

7 Multiplying and Dividing Fractions

Contrary to the fairly complicated process of adding and subtracting fractions, multiplying fractions together is an easy two step process.

1. Multiply the numerators together to get the numerator of the answer, and do the same to the denominator,
2. Reduce to lowest terms.

For example, let's multiply $\frac{5}{8} * \frac{2}{3}$:

1. We multiply the numerators and get 10; we multiply the denominators and get 24. Our fraction is therefore $\frac{10}{24}$.
2. We have a GCD of 2. Dividing out this GCD leaves $\frac{5}{12}$, which is the answer.

We would summarize this calculation as

$$\frac{5}{8} * \frac{2}{3} = \frac{10}{24} = \frac{5}{12}.$$

To divide fractions, we will flip the second fraction upside down and then multiply. For example, $\frac{7}{25}$ divided by $\frac{2}{5}$ is the same as $\frac{7}{25}$ times $\frac{5}{2}$. Multiplying these fractions gives $\frac{35}{50}$, and this reduces to $\frac{7}{10}$ after canceling out the common factor of 5.

Again, a summary:

$$\frac{7}{25} \div \frac{2}{5} = \frac{7}{25} * \frac{5}{2} = \frac{35}{50} = \frac{7}{10}.$$

8 From Integers to Polynomials

We want to do the same things with polynomial that we do with integers: add, subtract, multiply, divide with remainder, factor, find GCDs and LCMs, and manipulate fractions.

Most of the above holds true even with polynomials. In fact, the only essential difference is that the factoring step is generally much harder. In particular, the algorithms outlines above for finding GCDs, finding LCMs, and manipulating fractions still work if we replace the integers with polynomials.

An example problem: Find

$$\frac{2}{(x+2)^2(x-1)} - \frac{6}{(x+2)(x-1)^2}.$$

Solution: The two denominators have an LCM of $(x{+}2)^2(x{-}1)^2$. Converting the two fractions to this denominator, we have

$$\frac{2(x-1)}{(x+2)^2(x-1)^2} - \frac{6(x+2)}{(x+2)^2(x-1)^2}$$

$$\frac{2x-2}{(x+2)^2(x-1)^2} - \frac{6x+12)}{(x+2)^2(x-1)^2}.$$

Now, add the numerators:

$$\frac{(2x-2)-(6x+12)}{(x+2)^2(x-1)^2}$$

$$\frac{-4x-14}{(x+2)^2(x-1)^2}.$$

The denominator is already factored, so let's factor the numerator to see if any common factors exist:

$$\frac{-2(2x+7)}{(x+2)^2(x-1)^2}.$$

Nope, no common factors (the GCD of the numerator and the denominator is 1), so this is the final answer.

For a final example, consider the multiplication problem

$$\frac{(x+2)(x-3)^3}{(x^2+1)^2(x-1)(x+2)^2} * \frac{2x^3+4x^2+2x+4}{3x^2-18x+27}.$$

We'll start by factoring the second fraction:

$$\frac{(x+2)(x-3)^3}{(x^2+1)^2(x-1)(x+2)^2} * \frac{2(x^3+2x^2+x+2)}{3(x^2-6x+9)}$$

$$\frac{(x+2)(x-3)^3}{(x^2+1)^2(x-1)(x+2)^2} * \frac{2((x+2)x^2+(x+2))}{3(x-3)^2}$$

$$\frac{(x+2)(x-3)^3}{(x^2+1)^2(x-1)(x+2)^2} * \frac{2(x+2)(x^2+1)}{3(x-3)^2}.$$

Now, multiply the numerators and denominators together:

$$\frac{2(x+2)^2(x-3)^3(x^2+1)}{3(x^2+1)^2(x-1)(x+2)^2(x-3)^2}$$

The GCD of the numerator and denominator is $(x+2)^2(x-3)^2(x^2+1)$, so we divide that out of both:

$$\frac{2(x-3)}{3(x^2+1)(x-1)}.$$

This completes the problem