Overanalysis of a Notational Quirk

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1 Introduction to the problem

To the dismay of students everywhere who are experiencing their first introduction to algebra, we use the same notation to describe a real number's reciprocal and a function's inverse. Indeed, most teachers of introductory algebra need to harp on their students that $f^{-1}(x) \neq \frac{1}{f(x)}$.

I started down this road in 2001 with an extra credit problem from my advanced calculus course: "Can you think of a function whose inverse equals its reciprocal?" I worked on the problem for about two weeks and found a few simple examples as well as a couple of properties such a function must have. I have since returned to this question three times over the past $4\frac{1}{2}$ years.

This write-up is the summary of the most interesting results that I discovered.

This article will deal only with functions from $\mathbb{R} \to \mathbb{R}$, with particular emphasis placed on functions from $\mathbb{R}^+ \to \mathbb{R}^+$.

2 Functions from $\mathbb{R} \to \mathbb{R}$

We shall begin by deriving a small set of useful formulas which will aid us in finding these functions. In order to do this, we will need the following lemma.

Lemma 1. *If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f^{-1}(x) = \frac{1}{f(x)}$ then*

1. *f is invertible*
2. *f is bijective*
3. *The domain of f is the same as its range*
4. *0 is not in the domain or range of f*

Proof. (1) We are specifying that $f^{-1}(a) = \frac{1}{f(a)}$, implying that f^{-1} is well-defined. (2) If f is invertible, then it is bijective. (3) If a is in the domain of f, then we know that we can plug a into f^{-1}. Thus a is in the domain of f^{-1}, which means a is in the range of f. (4) If 0 is in the domain of f, then 0 is also in its range, implying $f^{-1}(x) = \frac{1}{0}$ for some x–an impossibility. Therefore 0 is outside the domain of f, and is also outside its range. □

These next two propositions yield useful formulae to assist in analyzing this class of functions.

Proposition 1. $f^{-1}(x) = \frac{1}{f(x)} \Leftrightarrow f$ *is invertible and* $f(f(x)) = \frac{1}{x}$

Proof. ($\Rightarrow$) Starting with $f^{-1}(x) = \frac{1}{f(x)}$, we have f invertible from Lemma 1. Applying f to both sides, we get $f(f^{-1}(x)) = f(\frac{1}{f(x)})$. Simplifying, and applying f again, we get $f(x) = f(f(\frac{1}{f(x)}))$. Now, let $y = \frac{1}{f(x)}$. Then we have $\frac{1}{y} = f(f(y))$, and y in the range of f^{-1}. Thus, y is in the domain and range of f and therefore $f(f(x)) = \frac{1}{x}$ for all x in the domain of f.

($\Leftarrow$) Let $f(x) = y$. Then $f^{-1}(y) = x$ since f is invertible. Plugging this into $f(f(x)) = \frac{1}{x}$ yields $f(y) = \frac{1}{f^{-1}(y)}$, or $f^{-1}(x) = \frac{1}{f(x)}$. □

Proposition 2. $f^{-1}(x) = \frac{1}{f(x)} \Rightarrow \frac{1}{f(x)} = f(\frac{1}{x})$

Proof. From proposition 1, $f(f(x)) = \frac{1}{x}$. Apply f^{-1} to both sides to get $f(x) = f^{-1}(\frac{1}{x})$. However, $f^{-1}(\frac{1}{x}) = \frac{1}{f(\frac{1}{x})}$. Therefore, $f(x) = \frac{1}{f(\frac{1}{x})}$. Taking the reciprocal of both sides, $\frac{1}{f(x)} = f(\frac{1}{x})$. □

Corollary 1. *If* $f^{-1}(x) = \frac{1}{f(x)}$, *and 1 is in the domain of* f, *then* $f(1) = \pm 1$. *Similarly, if -1 is in the domain of* f, *then* $f(-1) = \pm 1$.

Proof. Plug $x = 1$ into the formula given by proposition 2: $\frac{1}{f(1)} = f(1)$. Solving, we get $f(1) = \pm 1$. □

We are now in a position to create a few simple examples of functions for which $f^{-1}(x) = \frac{1}{f(x)}$.

Example 1. $f : \{1\} \to \{1\}$ *defined by* $f(1) = 1$.

That's a pretty pathetic example. Here's a slightly less pathetic example.

Example 2. $f : \{-1, 1\} \to \{-1, 1\}$ *defined by* $f(1) = -1$ *and* $f(-1) = 1$.

For $x \neq \pm 1$, we will have $f(f(f(f(x)))) = x$ and $f(f(x)) = \frac{1}{x} \neq x$, and therefore all elements of the domain of f, other than ± 1 will have an orbit of length 4.

Example 3. $f : \{a, b, \frac{1}{a}, \frac{1}{b}\} \to \{a, b, \frac{1}{a}, \frac{1}{b}\}$, *where* $a, b \neq \pm 1$ *defined by*

$$f(x) = \begin{cases} b & x = a \\ \frac{1}{a} & x = b \\ \frac{1}{b} & x = \frac{1}{a} \\ a & x = \frac{1}{b} \end{cases}$$

There will be more interesting functions shortly; first we need a lemma.

Lemma 2. *Let f be continuous on an interval I. If f is injective, then f is strictly monotone.*

Proof. We shall prove this by contrapositive. Therefore, assume f is continuous on I but not strictly monotone. Then there exist $a, b, c \in I$ with $a < b < c$ such that one of two cases applies: (1) $f(a) < f(b)$ and $f(c) < f(b)$ (2) $f(b) < f(a)$ and $f(b) < f(c)$ Assume case (1) applies. Look at $[f(a), f(b)]$ and $[f(c), f(d)]$. These two sets have a non-empty intersection, so pick $y \in [f(a), f(b)) \cap [f(c), f(b))$. We have $y \in [f(a), f(b)]$, so by the Intermediate Value Theorem, $\exists e \in [a, b]$ such that $f(e) = y$; moreover, $y \neq f(b)$ so $e \neq b$. Similarly, $y \in [f(c), f(b)]$, so by the Intermediate Value Theorem, $\exists e' \in [b, c]$ such that $f(e') = y$; moreover, $y \neq f(b)$ so $e' \neq b$. Of course, $e \neq e'$, but $f(e) = f(e')$ so f is not injective. The proof for case (2) is nearly identical. $\square$

Here is one way to make a function for which $f^{-1} = \frac{1}{f}$: Start with a bijective function $f : \mathbb{R}^+ \to \mathbb{R}^-$ for which $f(1) = -1$. Find that function $g : \mathbb{R}^- \to \mathbb{R}^+$ which is f's inverse. Lastly, extend f to the negative reals by defining $f(x) = \frac{1}{g(x)}$ for $x < 0$.

Example 4. $g : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ *defined by*

$$g(x) = \begin{cases} -x & x < 0 \\ -\frac{1}{x} & x > 0 \end{cases}$$

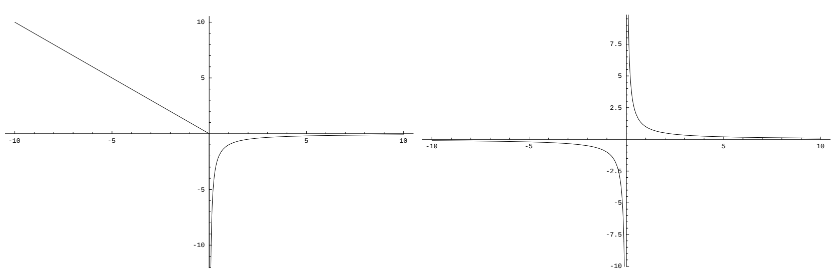

Figure 1: $g(x)$ and $g(g(x))$

Example 5. $h : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ *defined by*

$$h(x) = \begin{cases} \frac{\ln(2)}{\ln(1-x)} & x < 0 \\ 1 - 2^x & x > 0 \end{cases}$$

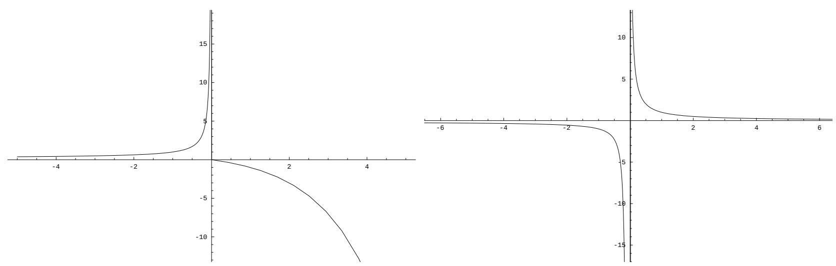

Figure 2: $h(x)$ and $h(h(x))$

3 Functions from $\mathbb{R}^+ \to \mathbb{R}^+$

While examples 4 and 5 give examples of very continuous functions for which $f^{-1} = \frac{1}{f}$. If we restrict ourselves to $\mathbb{R}^+$ it becomes much harder. In fact, proposition 4 will guarantee at least one discontinuity, while the propositions following will force still more discontinuities.

Proposition 3. *If* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies* $f^{-1}(x) = \frac{1}{f(x)}$ *then*

1. $\lim_{x\to 0} f(x) \neq 0$
2. $\lim_{x\to 0} f(x) \neq \infty$
3. $\lim_{x\to \infty} f(x) \neq 0$
4. $\lim_{x\to \infty} f(x) \neq \infty$

Proof. (1) If $\lim_{x\to 0} f(x) = 0$ then $\lim_{x\to 0} f(f(x)) = 0$. But $\lim_{x\to 0} f(f(x))$ should equal ∞ by $f(f(x)) = \frac{1}{x}$. (2) If $\lim_{x\to 0} f(x) = \infty$, then by $f(f(x)) = \frac{1}{x}$ we get $\lim_{x\to\infty} f(x) = \infty$. But $\lim_{x\to\infty} f(x) = \infty$ implies $\lim_{x\to\infty} f(f(x)) = \infty$.

This contradicts $f(f(x)) = \frac{1}{x}$. (3) We know that $\lim_{x\to\infty} f(f(x)) = 0$, so if $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to 0} f(x) = 0$. That can't happen by (1). (4) We proved this in route to proving (2). □

Proposition 4. *If $f(1) = 1$ and $f^{-1}(x) = \frac{1}{f(x)}$, then either f is discontinuous at $x = 1$ or f has discontinuities arbitrarily close to $x = 1$.*

Proof. Assume by way of contradiction that $f(1) = 1$ and that f is continuous on the interval $(1-\epsilon, 1+\epsilon)$ with $\epsilon > 0$. By Lemma 2, f must be monotone. Assume f is increasing. We can find a $u > 1$ such that $f(u) \in (1, 1+\epsilon)$. $f(f(u))$ is then greater than 1. But $f(f(u)) = \frac{1}{u} < 1$, which creates a contradiction. Now assume f is decreasing. Then we can find a $u < 1$ so that $f(u) \in (1, 1+\epsilon)$. $f(f(u))$ would then be less than 1. But $f(f(u)) = \frac{1}{u} > 1$. This is also a contradiction. □

Can $f(x)$ be continuous at $x = 1$? Yes, and I have an example showing this which appears at the end. For now, here is an example which is discontinuous at $x = 1$.

Example 6. $f_1 : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by*

$$f_1(x) = \begin{cases} 2 - 2x & 0 < x < \frac{1}{2} \\ 1 - \frac{1}{2x} & \frac{1}{2} < x < 1 \\ 1 & x = 1 \\ \frac{2}{2-x} & 1 < x < 2 \\ \frac{x}{2x-2} & x > 2 \end{cases}$$

How did I come up with this example? Well, first I chose the points $\frac{1}{2}$, 1 and 2 to be discontinuities, then took a linear mapping of $(0, \frac{1}{2})$ to $(1, 2)$ (notice that proposition 3 is not violated). Since $f(f(x)) = \frac{1}{x}$, I found the inverse of the linear mapping, and took its reciprocal. That gave me a mapping from $(1, 2)$ to $(2, \infty)$, so I found the inverse of that and took its reciprocal. Natually, that was a mapping from $(2, \infty)$ to $(\frac{1}{2}, 1)$. Repeating this procedure, we get the final piece; a mapping from $(\frac{1}{2}, 1)$ to $(0, \frac{1}{2})$. Define $f(1) = 1$, and piece these mappings together.

You'll no doubt notice that we have a function with three discontinuities. Unfortunately, f_1 is not defined at $x = \frac{1}{2}$ or $x = 2$. Moreover, if we attempt to define f_1 at these points while retaining $f_1^{-1}(x) = \frac{1}{f_1(x)}$, we require f_1 to remain bijective. Thus $f_1(2)$ must be either $\frac{1}{2}$ or 2. However, if $f_1(2) = 2$, then $f_1(f_1(2)) = f_1(2) = 2 \neq \frac{1}{2}$. And if $f_1(2) = \frac{1}{2}$, then $f_1(\frac{1}{2}) = f_1(f_1(2)) = \frac{1}{2} = f_1(2)$, so f_1 fails to be bijective. Thus we cannot define f_1 and retain $f_1^{-1}(x) = \frac{1}{f_1(x)}$. Another way to see that f_1 cannot be extended to these points is that we have exactly two points at which f_1 is not defined, both of which are not ± 1. Thus these points must form orbits of length 4, which is greater then the number of points availible. Strictly speaking, then, f_1 maps the set $\mathbb{R}^+ - \{\frac{1}{2}, 2\}$ to itself.

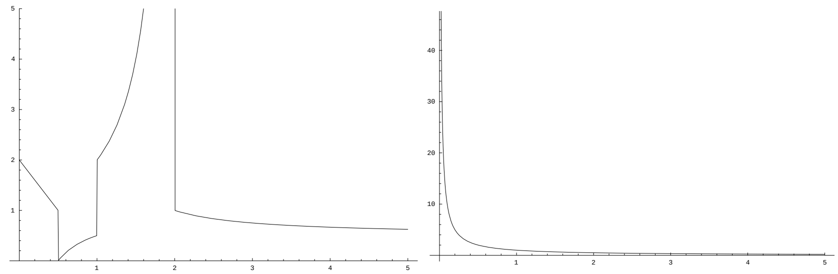

Figure 3: $f_1(x)$ and $f_1(f_1(x))$

In fact, if $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f^{-1}(x) = \frac{1}{f(x)}$ and we require f to be defined over all of $\mathbb{R}^+$ then f must have infinitely many dicsontinuities. In order to prove this, we will need a few lemmas.

Lemma 3. *If $f^{-1}(x) = \frac{1}{f(x)}$ and f is discontinuous at $x = a$ then f is discontinuous at $x = \frac{1}{a}$.*

Proof. $f(x)$ continuous at $x = a \Rightarrow \frac{1}{f(x)}$ continuous at $x = a \Rightarrow f(\frac{1}{x})$ continuous at $x = a \Rightarrow f(x)$ continuous at $x = \frac{1}{a} \Rightarrow \frac{1}{f(x)}$ continuous at $x = \frac{1}{a} \Rightarrow f(\frac{1}{x})$ continuous at $x = \frac{1}{a} \Rightarrow f(x)$ continuous at $x = a$. □

Corollary 2. *If $f^{-1}(x) = \frac{1}{f(x)}$, $f : \mathbb{R}^+ \to \mathbb{R}^+$ and f has n discontinuities, then $n \equiv 1 (mod\ 2)$.*

Proof. Since $n < \infty$, f is discontinuous at $x = 1$. Any other point a at which f is discontinuous implies another discontinuity at f at $\frac{1}{a}$. □

4 Foray Into Topology

Before we prove that $f : \mathbb{R} \to \mathbb{R}$ with $f^{-1}(x) = \frac{1}{x}$ requires f to have infinitely many discontinuities, we'll need to prove a technical lemma. In order to prove the technical lemma, there are three topological facts that are required.

Lemma 4. *The image of a connected set under a continuous map is connected.*

Proof. (Lifted from Munkres, p. 150). Let $f : X \to Y$ be continuous and X connected. We may assume that f is surjective since we are only concerned with the image. Suppose that $Y = A \cup B$ is a separation of Y into disjoint nonempty open sets. Then $f^{-1}(A)$ and $f^{-1}(B)$ are open since f is continuous, disjoint, and nonempty since f is surjective. Thus $X = f^{-1}(A) \cup f^{-1}(B)$ is not connected. □

Lemma 5. *Let $f : I \to J$ be a continuous, strictly monotone map where I and J are intervals in $\mathbb{R}$. Then f is an open map.*

Proof. Let $K = (a, b)$ be an interval contained in I. Then, given any $c \in K$, $f(c)$ lies strictly between $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$. Moreover, by continuity and the intermediate value theorem, given any d between these two bounds, we can find e such that $f(e) = d$. Therefore the image is an open interval. □

Lemma 6. *If A and B are intervals in $\mathbb{R}$ and $f : A \to B$ has an inverse, then f is continuous on $A \Leftrightarrow f^{-1}$ is continuous on B.*

Proof. Assume f is continuous. Since f is invertible, it is strictly monotone by lemma 2. Therefore the previous lemma applies and f is an open map. Since f is an open map, f^{-1} is continuous. The converse holds by symmetry. □

5 Back to the Main Program

These three items are required to prove this technical lemma.

Lemma 7. *Let $f^{-1}(x) = \frac{1}{f(x)}$, $f : \mathbb{R}^+ \to \mathbb{R}^+$, and $D = \{a_1, a_2, \cdots, a_n\}$ be the set of points at which f is discontinuous ($a_1 < a_2 < \cdots < a_n$). Define $a_0 = 0$ and $a_{n+1} = \infty$. Then given $0 \le k \le n$, f maps the interval (a_k, a_{k+1}) to the interval (a_m, a_{m+1}) for some $0 \le m \le n$.*

Proof. By hypothesis, f is continuous on the connected set $I_k = (a_k, a_{k+1})$. Thus, $f(I_k)$ is also connected. Since f is invertible, f must be strictly monotone on (a_k, a_{k+1}), and therefore $f(I_k)$ is open. Since $f(I_k)$ is open and connected, we may write it as (b_1, b_2) for some nonnegative b_1, b_2. We automatically get f^{-1} continuous on (b_1, b_2), and therefore $f = \frac{1}{f^{-1}}$ is continuous on (b_1, b_2). Therefore, (b_1, b_2) contains no discontinuous points of f; i.e. $\forall 0 \le j \le n, a_j \notin (b_1, b_2)$. This implies $(b_1, b_2) \subseteq (a_m, a_{m+1}) = I_m$ for some $0 \le m \le n$. f is continuous on I_m, so $f^{-1} = \frac{1}{f}$ is continuous on I_m. Therefore $f^{-1}(I_m)$ is an open connected set; call it (c_1, c_2). Moreover, $I_k = f^{-1}((b_1, b_2)) \subseteq f^{-1}(I_m) = (c_1, c_2)$. Of course, we have f continuous on (c_1, c_2), so $c_1 = a_k$ and $c_2 = a_{k+1}$. Finally, $f(I_k) = f((c_1, c_2)) = f(f^{-1}(I_m) = I_m$ as required. □

Corollary 3. *The intervals in lemma 4 have orbits of length 4 under f.*

Proof. This follows immediately by noting that the intervals are disjoint, and each individual point in the interval has an orbit of length 4. □

Now we have enough to prove the interesting result–infinitely many discontinuities are required if we map positive reals to positive reals.

Theorem 1. *If $f^{-1}(x) = \frac{1}{f(x)}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$, then f has infinitely many discontinuities.*

Proof. Assume by way of contradiction that n is the number of discontinuities and that n is finite. From corollary 2, $n \equiv 1 (\text{mod } 2)$. We either have $n \equiv 1 (\text{mod } 4)$ or $n \equiv 3 (\text{mod } 4)$. Assume first that $n \equiv 1 (\text{mod } 4)$. Then $n = 4m+1$ for some integer m. We then have $2m$ discontinuities in the interval $(0, 1)$, $2m$ discontinuities in the interval $(1, \infty)$ and 1 discontinuity at 1. Label the points of discontinuity in $(0, 1)$ by $\{a_1, a_2, \cdots, a_{2m}\}$ so that $a_1 < a_2 < \cdots < a_{2m}$. Now, look at our set of continuous intervals: $\{(0, a_1), (a_1, a_2), \cdots, (a_{2m-1}, a_{2m}), (a_{2m}, 1), (1, \frac{1}{a_{2m}}), \cdots, (\frac{1}{a_2}, \frac{1}{a_1}), (\frac{1}{a_1}, \infty)\}$. This set has $4m + 2$ elements. The orbits of these intervals have length 4. Thus we cannot have $f^{-1} = \frac{1}{x}$, owing to the two leftover intervals. Now assume that $n \equiv 3 (\text{mod } 4)$. Then $n = 4m + 3$ for some integer m. We know that 1 is a discontinuous point, but we also know that $f(1) = 1$. All the other points of discontinuity have orbits of length 4, but we have $4m + 2$ of these points left after taking care of 1. Thus we cannot have $f^{-1} = \frac{1}{f}$, owing to the two leftover points. □

The moment has come to put forth an function f for which $f^{-1} = \frac{1}{f}$ which is defined for every positive real number. This will also be an example of a function which is continuous at 1. Take as the set of discontinuities the set of real numbers of the form $\frac{k}{k+1}$ where k is a positive integer. By lemma 3, this will force discontinuities at each $\frac{k+1}{k}$. Now, for each odd integer n, define f to be an increasing linear mapping from $(\frac{n-1}{n}, \frac{n}{n+1}]$ to $(\frac{n}{n+1}, \frac{n+1}{n+2}]$. From here, solve for the other three intervals of the orbit of $(\frac{n-1}{n}, \frac{n}{n+1})$ as we did in example 6. The net result appears below.

Example 7. *For each positive odd integer n, define:*

$$f(x) = \begin{cases} \frac{(n+1)nx+2n+1}{(n+1)(n+2)} & x \in (\frac{n-1}{n}, \frac{n}{n+1}] \\ \\ \frac{n(n+1)}{(n+1)(n+2)x-2n-1} & x \in (\frac{n}{n+1}, \frac{n+1}{n+2}] \\ \\ \frac{\frac{(n+1)(n+2)}{x}-2n-1}{n(n+1)} & x \in [\frac{n+2}{n+1}, \frac{n+1}{n}) \\ \\ \frac{(n+1)(n+2)}{\frac{n(n+1)}{x}+2n+1} & x \in [\frac{n+1}{n}, \frac{n}{n-1}) \end{cases}$$

Further, if $n = 1$ then replace $\frac{n}{n-1}$ with ∞. Finally, define $f(1) = 1$.

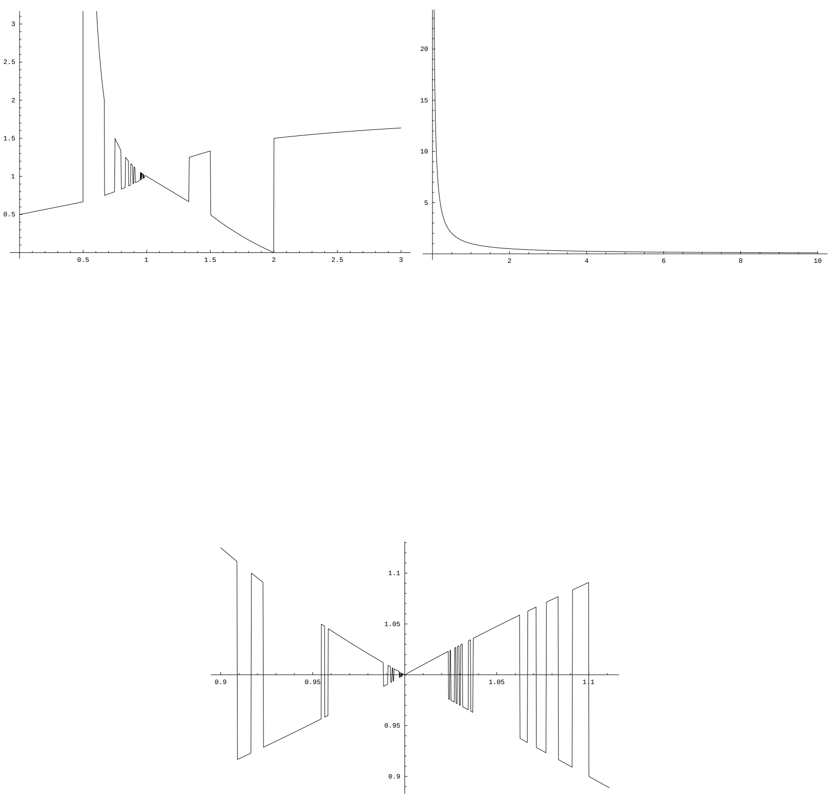

Figure 4: $f(x)$ on $(0, 5)$, $f(x)$ on $(\frac{9}{10}, \frac{10}{9})$, and $f(f(x))$ on $(0, 5)$

6 References

[1] Munkres, James R.: *Topology*, 2nd Edition, Prentice Hall, 2000, p. 150.

[2] MacKendrick, Sharon: *For What Functions Is* $f^{-1}(x) = \frac{1}{f(x)}$ *?*, College Mathematics Journal **34** (September 2003), pp.304-311.