Theorem 1 (Riesz Representation Theorem) *Let H be a Hilbert space and let $\Lambda \in H^*$ be given. Then, $\exists!\ z_0 \in H$ such that $\Lambda x = (x, z_0)\ \forall x \in H$, and $\|\Lambda\| = \|z_0\|$.*

Proof:

Let $N = N_\Lambda = \{y \in H\ s.t.\ \Lambda y = 0\}$. If $N = H$, then $\Lambda x = (x, 0)\ \forall x \in H$ so we can in that case take $z_0 = 0$. So suppose $N \neq H$.

N is closed. Indeed, if $\{y_n\} \in N$ and $y_n \to y$, then by continuity of Λ, $\Lambda y_n \to \Lambda y$. So $\Lambda y = lim_{n\to\infty} \Lambda y_n = lim_{n\to\infty} 0 = 0$, i.e. $y \in N$. $\therefore$ by projection theorem, $\exists z_1 \in N^\perp$, $z_1 \neq 0$. We simply take any $x_1 \in H - N$, and then $x_1 = y_1 + z_1 \in N \oplus N^\perp$. $z_1 \neq 0$, since $y_1 \in N$, $x_1 \notin N$.

Claim 2 *$dim N^\perp = 1$.*

Take any $0 \neq w_1, w_2 \in N^\perp$. Then $\Lambda w_1 \neq 0$ and $\Lambda w_2 \neq 0$, both are $\in \mathbb{C}$. Since $dim\mathbb{C} = 1$, $\exists \alpha_1, \alpha_2$ not both equal to zero, such that $\alpha_1 \Lambda w_1 + \alpha_2 \Lambda w_2 = 0$. But by linearity of Λ, this implies $\Lambda(\alpha_1 w_1 + \alpha_2 w_2) = 0$. Therefore $\alpha_1 w_1 + \alpha_2 w_2 \in N$, but also since $N^\perp$ is a vector space, $\alpha_1 w_1 + \alpha_2 w_2 \in N^\perp$. This is only possible if $\alpha_1 w_1 + \alpha_2 w_2 = 0$. Such α_1, α_2 exist $\forall$ such w_1, $w_2 \in N^\perp \Longrightarrow dim N^\perp = 1$.

We know $\Lambda x \in \mathbb{C}$, $\forall x \in H$. Thereby, take $\Lambda z_1 = (z_1, \alpha z_1)$, and solve for α. We get

$$\alpha = \overline{\frac{\Lambda z_1}{(z_1, z_1)}}$$

Letting $\alpha z_1 = z_0$, we get $\Lambda z_1 = (z_1, z_0)$. Then since $dim N^\perp = 1$ and by linearity in the first position of the scalar product, we get that $\Lambda z = (z, z_0)$, $\forall z \in N^\perp$. Also, for any $y \in N$, $0 = \Lambda y = (y, z_0)$, since $z_0 = \alpha z_1 \in N^\perp$.

Now by the projection theorem again, $x \in H \Longrightarrow x = y + z \in N \oplus N^\perp$. So $\Lambda x = \Lambda(y + z) = \Lambda y + \Lambda z = (y, z_0) + (z, z_0) = (y + z, z_0) = (x, z_0)$.

Now suppose there is some $z_0^{'}$ with the same properties as z_0. Then $0 = \Lambda_{z_0} x - \Lambda_{z_0^{'}} x = (x, z_0 - z_0^{'})$, for any $x \in H$, in particular for $x = z_0 - z_0^{'}$. Therefore $\|z_0 - z_0^{'}\|^2 = 0 \Longrightarrow z_0 - z_0^{'} = 0$ and therefore $z_0 = z_0^{'}$, so z_0 is unique.

Finally, we use the Schwarz inequality to show $\|\Lambda\| = \|z_0\|$:

$$\|\Lambda\| = \sup_{\|x\|=1} |\Lambda x| = \sup_{\|x\|=1} |(x, z_0)| \leq \sup_{\|x\|=1} \|x\|\ \|z_0\| = \|z_0\|$$

and

$$\|z_0\|^2 = |(z_0, z_0)| = |\Lambda z_0| \leq \|\Lambda\|\ \|z_0\| \iff \|z_0\| \leq \|\Lambda\|$$

Therefore, $\|\Lambda\| = \|z_0\|$.

QED.