Additional Notes – Optimization (Section 3.4)

(In class, I have done things for optimization slightly differently than the book, so I'll add these notes about it here:)

We'll consider a function $f(x)$ which is continuous on an interval. The interval may look like

$$[a, b], \qquad [a, b), \qquad (a, b] \quad or \quad (a, b),$$

that is, the interval may be closed (furthest left, above), partly open, or open (furthest right, above). Remember, this means x is in the above intervals when

$$a \leq x \leq b, \qquad a \leq x < b, \qquad a < x \leq b \quad or \quad a < x < b,$$

respectively.

Whenever an endpoint is "open", that is, when there is a round bracket next to it, I'll allow for the case when a or b is $\pm\infty$. For example, the cases above allow for

$$(-\infty, 5], \qquad (-3, \infty), \qquad \text{etc}\ldots$$

meaning,

$$x \leq 5 \quad or \quad x > -3, \qquad \text{etc}\ldots$$

So for example, I could consider the function $f(x) = \frac{1}{x}$ on the interval $(0, 1]$ or $(-\infty, 0)$, as it is continuous for $0 < x \leq 1$ and for $x < 0$.

So that is the setup. Now I'll give the question and method of answering it when we are in any of the situations above:

Question: What is the absolute maximum or absolute minimum of f over the interval considered, and where is it attained, or is there possibly no maximum or no minimum?

(continued ⟶)

Method:

1. Take the derivative of f, wherever it exists.
2. Find all points x between a and b such that *either*:
   (a) $f'(x) = 0$, or
   (b) $f'(x)$ is not defined or does not exist.

   Let's call all these x-values, as well as the endpoints of the interval, a and b (whatever they are, two numbers including possibly $-\infty$ or ∞), the "critical points".
3. Evaluate f at all critical points, and at all endpoints with a "closed" bracket. For all endpoints with an "open" bracket, including possibly $a = -\infty$ or $b = \infty$, evaluate

$$\lim_{x \to a^+} f(x) \qquad \text{or} \qquad \lim_{x \to b^-} f(x).$$

   For example, if we consider the interval $[a, b)$, and $f'(x)$ is either zero or undefined at, say, x_1, x_2 and x_3 which are all (and are the only such values) between a and b, then we would want to calculate the numbers (possibly infinite)

$$f(x_1), \qquad f(x_2),$$

$$f(x_3), \qquad f(a),$$

   and

$$\lim_{x \to b^-} f(x),$$

   giving us, in that case, five values.

   Let us call all these values the "critical values".
4. **(Answer:)**

   If the largest critical value occurs at an open endpoint, then f has no absolute maximum on the interval. Otherwise, the critical point where f takes the highest value is where f attains its absolute maximum on the interval, and the largest critical value is the maximum of f.

   Similarly, if the smallest critical value occurs at an open endpoint, then f has no absolute minimum on the interval. Otherwise, the critical point where f takes the highest value is where f attains its absolute minimum on the interval, and the lowest critical value is the minimum of f.

Remember: $-\infty$ or ∞ is only ever included as an "open" endpoint. So in the last step, this largest value may occur "at ∞" or "at $-\infty$".

(examples $\longrightarrow$)

Ex. 1 Discuss the absolute extrema of $f(x) = \frac{x^2}{4-x^2}$ on the interval $(-1, 2)$.

1. Taking the derivative, we find
$$f'(x) = \frac{8x}{(4-x^2)^2}$$

2. $f'(x) = 0$ when $x = 0$, and $f'(x)$ is undefined when $x = 2$ or $x = -2$, so the only critical points in the interval $(-1, 2)$ are -1, 0 and 2.

3. We compute:
$$\lim_{x\to(-1)^+} f(x) = \frac{1}{3},$$
$$f(0) = 0$$
$$\lim_{x\to 2^-} f(x) = +\infty$$
Therefore the critical values are $\frac{1}{3}$, 0 and $+\infty$.

4. The largest critical value, $+\infty$, occurs at the open endpoint 2. Therefore f has no absolute maximum on $(-1, 2)$. (If you look at the graph of the function, you'll see that the function keeps going up to infinity as x becomes very close to 2 from the left.)

   The smallest critical value, 0, occurs at the critical point $x = 0$, which is not an open endpoint (it's not an endpoint at all). Therefore 0 is the absolute minimum of f on $(-1, 2)$, and occurs at $x = 0$.

If you graph the function, you will see that this makes sense.

Ex. 2 Find the largest and smallest value of $f(x) = \frac{1}{x}$ when $x \leq -2$.

1. $$f'(x) = -\frac{1}{x^2}$$

2. $f'(x)$ is undefined when $x = 0$ and is never equal to zero, and 0 is not less than -2. So our only critical points are the endpoints: $-\infty$ and -2. ($-\infty$ is the left endpoint!!! Our interval is $(-\infty, -2]$).

3. $$\lim_{x\to-\infty} f(x) = 0$$
$$f(-2) = -\frac{1}{2}$$
0 and $-\frac{1}{2}$ are the critical values.

4. The largest critical value is attained at an open endpoint ($-\infty$), so the function has no maximum in the interval. There is an absolute minimum of $-\frac{1}{2}$ at $x = -2$.

   So, there is no largest value. The smallest value is $-\frac{1}{2}$.