

**Principle of Uniform Boundedness 1** *Let  $H$  be a Hilbert space, with a family  $\{T_\alpha\} \subset \mathcal{B}(H)$ . If the set  $\{T_\alpha x\}$  is bounded for each  $x \in H$ , then the set  $\{\|T_\alpha\|\}$  is bounded.*

Proof:

Suppose there exists a ball  $B(x_0, \epsilon)$  in which  $\{T_\alpha x\}$  is uniformly bounded, i.e. there exists some  $K > 0$  such that  $\|T_\alpha x\| \leq K$  for any  $\alpha$ , if  $\|x - x_0\| < \epsilon$ . Then for any  $y \neq 0$  we can consider  $z = \frac{\epsilon}{2\|y\|}y + x_0$ , and we have

$$\|z - x_0\| = \left\| \frac{\epsilon}{2\|y\|}y + x_0 - x_0 \right\| = \left\| \frac{\epsilon}{2\|y\|}y \right\| = \frac{\epsilon\|y\|}{2\|y\|} = \frac{\epsilon}{2}$$

So by assumption,  $\|T_\alpha z\| \leq K$  for all  $\alpha$ .

But

$$\begin{aligned} \|T_\alpha z\| &= \left\| T_\alpha \left( \frac{\epsilon}{2\|y\|}y + x_0 \right) \right\| \\ &= \left\| \frac{\epsilon}{2\|y\|}T_\alpha y - (-T_\alpha x_0) \right\| \\ &\geq \frac{\epsilon}{2\|y\|} \|T_\alpha y\| - \| -T_\alpha x_0 \| \\ &= \frac{\epsilon}{2\|y\|} \|T_\alpha y\| - \|T_\alpha x_0\| \end{aligned}$$

$$\therefore \frac{\epsilon}{2\|y\|} \|T_\alpha y\| - \|T_\alpha x_0\| \leq \|T_\alpha z\| \leq K$$

Then we also have

$$\frac{\|T_\alpha y\|}{\|y\|} \leq \frac{2(K + \|T_\alpha x_0\|)}{\epsilon} \leq \frac{2(K + \sup_\alpha \|T_\alpha x\|)}{\epsilon} = \frac{2(K + K')}{\epsilon}$$

We know that  $K' = \sup_\alpha \|T_\alpha x_0\| < \infty$  by our assumption since  $x_0 \in B(x_0, \epsilon)$ . Therefore

$$\|T_\alpha\| = \sup_{y \neq 0} \frac{\|T_\alpha y\|}{\|y\|} \leq \frac{2(K + K')}{\epsilon}$$

for any  $\alpha$ . Thus  $\{\|T_\alpha\|\}$  is bounded.

Now we prove that there does exist such a ball. Suppose there does not exist any  $x_0 \in H$ ,  $\epsilon, K > 0$  such that  $\|T_\alpha x\| \leq K$  for  $\|x - x_0\| < \epsilon$ , but that the assumption of our theorem still holds, i.e. for any  $x$ , there is some  $K > 0$  such that  $\|T_\alpha x\| \leq K$  for all  $\alpha$ .

Take any ball, call it  $B_0$ . Then there exists an  $\alpha_1$  such that  $\{\|T_{\alpha_1} x\|\}$  is unbounded for  $x \in B_0$ , in particular  $\exists x_1 \in B_1$  such that  $\|T_{\alpha_1} x_1\| > 1$ . By continuity of  $T_{\alpha_1}$ ,  $\|T_{\alpha_1} x\| > 1$  for  $x \in B(x_1, \epsilon_1)$ , for some  $\epsilon_1 > 0$ . Take  $\epsilon_1$  small

enough that  $\epsilon_1 < 1$  and  $B(x_1, \epsilon_1) \subset B_0$ . Then again by the assumption of our theorem  $\exists x_2 \in B(x_1, \epsilon_1)$  such that  $\|T_{\alpha_2} x_2\| > 2$  for some  $x_2$  and again by continuity of  $T_{\alpha_2}$ ,  $\|T_{\alpha_2} x\| > 2$  for  $x \in B(x_2, \epsilon_2)$ . Take  $\epsilon_2$  such that  $\epsilon_2 < \frac{1}{2}$  and  $B(x_2, \epsilon_2) \subset B(x_1, \epsilon_1)$ .

Continuing in this fashion, we arrive at a sequence of balls such that  $B(x_n, \epsilon_n) \subset B(x_{n-1}, \epsilon_{n-1})$  and  $\epsilon_n < \frac{1}{n}$ , and  $\|T_{\epsilon_n} x\| > n$  for all  $x \in B(x_n, \epsilon_n)$ .

We now reference a lemma which we will state but not prove, from “Foundations of Modern Analysis,” by Avner Friedman, Theorem 3.4.1, which is a special case of the Baire category theorem:

**Lemma 2** *Let  $\{F_n\}$  be a monotone-decreasing sequence of non-empty closed sets, in a complete metric space. If the sequence of the diameters of the sets converges to zero, then there exists one and only one point that belongs to  $\bigcap_{n=1}^{\infty} F_n$ .*

Therefore since  $H$  can be made a complete metric space via the inner product norm, and since our sequence of sets  $\{B(x_n, \epsilon_n)\}$  satisfies the assumptions of the lemma, there exists a unique point  $z \in H$  such that

$$z \in \bigcap_{n=1}^{\infty} B(x_n, \epsilon_n)$$

But then  $\|T_{\alpha_n} z\| > n$  for any  $\alpha_n$ , so for any given  $N$  we can find  $n$  such that  $\|T_{\alpha_n} z\| > N$ , so that  $\|T_{\alpha_n} z\|$  is unbounded, contrary to our assumption.

Such a ball therefore exists, and our theorem is proved.