Assignment 1 due Wed Sept 19:

1.7, 1.9, 2.6, 2.13, 2.14, 2.15

Hints to Chapter 1:

1.7 It suffices to prove that $A^c \in \mathcal{F}^P$ if $P(A)=0$.

2.6 To prove (i)$\Longrightarrow$(ii) observe that, for every $k \geq 1$, in the $1/k$-neighborhood of a point from a $1/k$-net there are infinitely many elements of x_n, which allows one to choose a Cauchy subsequence. To prove (ii)$\Longrightarrow$(i) assume that for an $\varepsilon > 0$ there is no finite ε-net and find a sequence of $x_n \in K$ such that $\rho(x_n, x_m) \geq \varepsilon/3$ for all n, m.

2.10 Assume the contrary.

2.14 Observe that $N_b(n)$ is the number of $i = 1, ..., n$ such that $10^k b \leq 2^i < 10^k(b+1)$ for some $k = 0, 1, 2, ...$ and then take $\log_{10}$.

2.15 Define

$$\bar{f}(x) = \lim_{\varepsilon \downarrow 0} \sup_{y:|y-x|<\varepsilon} f(y), \quad \underline{f}(x) = \lim_{\varepsilon \downarrow 0} \inf_{y:|y-x|<\varepsilon} f(y)$$

and prove that $\Delta_f = \{\bar{f} \neq \underline{f}\}$ and the sets $\{x : \bar{f}(x) < c\}$ and $\{x : \underline{f}(x) > c\}$ are open.

3.3 Attached points $t_1, ..., t_n$ and n may vary.

3.4 Show that the set of all such A is a σ-field.

4.12 Let $\alpha^2 = E\xi^2$. Observe that $P(|\xi| \geq x) = P(|\xi/\alpha| \geq x/\alpha)$. Then in the integral $\int_{x/\alpha}^{\infty} \exp(-y^2/2)\, dy$ first replace α with σ and after that divide and multiply the integrand by y.

Grades obtained for Assignment 1:

35, 34, 33, 33, 32, 30, 29, 29, 28, 28, 28, 25, 24, 22, 21.

Assignment 2 due Wed Oct 3:

1.1.10 (ii), 1.3.3, 1.3.5, 1.4.11, 1.4.13, 1.4.14

Grades obtained for Assignment 2:

36, 36, 36, 36, 35, 35, 35, 35, 34, 34, 33, 33, 33, 29, 28.

Grades obtained for Assignments 1+2:

71, 69, 68, 68, 67, 65, 65, 64, 64, 61, 57, 57, 55, 55, 53.

Assignment 3 due Wed Oct 17:

2.2.5, 2.2.10, 2.2.11, 2.3.8, 2.3.14, 2.3.21.

Hints to Chapter 2:

2.5 Use Exercise 1.4.14, with $R(x) = x$ and estimate $\int_0^x \sqrt{(-\ln y)/y}\, dy$ through $\sqrt{x(-\ln x)}$ by using l'Hospital's rule.

2.10 The cases $a \leq b$ and $a > b$ are different.

2.12 If $P(\xi \leq a, \eta \leq b) = \int_{-\infty}^{b} f(x)\, dx$ for every b, then $Eg(\eta)I_{\xi \leq a} = \int_{\mathbb{R}} g(x)f(x)\, dx$. The result of these computations is given in Sec. 6.8.

3.4 It suffices to prove that the indicators of sets $(s, t]$ are in $L_p(\Pi, \mu)$.

3.8 Observe that

$$\varphi(s) = E\exp(i\sum_{n=1}^{\infty} f(s+\sigma_n))$$

and by using independence of τ_n and the fact that $EF(\tau_1,\tau_2,...) = E\Phi(\tau_1)$, where $\Phi(t) = EF(t,\tau_2,...)$, show that

$$\varphi(s) = \int_0^{\infty} e^{if(s+t)-t}\varphi(s+t)\,dt = e^s\int_s^{\infty} e^{if(t)}(e^{-t}\varphi(t))\,dt.$$

Conclude first that φ is continuous, then that $\varphi(s)e^{-s}$ is differentiable, and solve the above equation. After that approximate by continuous functions the function which is constant on each interval $(t_j, t_{j+1}]$ and vanishes outside of the union of these intervals.

3.14 Prove that, for every Borel nonnegative f, we have

$$E\sum_{\sigma_n\le 1} f(\sigma_n) = \int_0^1 f(s)\,ds$$

and use it to pass to the limit from step functions to arbitrary ones.

3.21 For $b_n > 0$ with $b_n \to 1$, we have $\prod b_n = 0$ if and only if $\sum_n (1-b_n) = \infty$.

3.23 Use Remark 1.4.10 and (3.6).

5.9 Take any continuous function $u(x)$ defined on $[a,b]$ such that $u < 0$ in (a,b) and $u(a) = u(b) = 0$ and use it to write a formula similar to (5.1).

6.7 Define τ as the first exit time of x_t from (a,b) and similarly to (6.3) prove that

$$u(0) = E\int_0^{\tau} e^{-\lambda t}(\lambda u(x_t) - u'(x_t) - (1/2)u''(x_t))\,dt + Ee^{-\lambda\tau}u(x_\tau).$$

7.7 Observe that $\int_{(0,t]} \pi_s\,d\pi_s = \pi_t(\pi_t+1)/2$.

7.8 First take $f_t = I_\Delta$.

8.4 Keep in mind the proof of Theorem 8.2 and redo Exercise 7.5 for π_t in place of w_t.

8.5 Take a sequence of step functions converging to f μ-a.e. and observe that step functions are $\mathcal{F}_t$-adapted.

Grades obtained for Assignment 3:

34, 34, 31, 31, 29, 29, 28, 28, 26, 25, 24, 23, 21, 21, 18.

Grades obtained for Assignments 1+2+3:

105, 99, 99, 97, 97, 96, 92, 92, 90, 84, 80, 79, 79, 78, 74.

Grades obtained for Midterm:

36, 35, 35, 34, 33, 32, 32, 32, 31, 31, 31, 25, 25, 20, 16.

Grades obtained for Assignments 1+2+3+Mdtr:

140, 135, 133, 131, 129, 124, 123, 122, 121, 117, 109, 106, 104, 100, 95.

Assignment 4 due Wed Oct 31:

2.3.23, 2.5.9 , 3.1.3, 3.1.17 , 3.2.3, 3.3.3

Grades obtained for Assignment 4:

36, 35, 33, 33, 33, 32, 32, 31, 31, 29, 29, 28, 27, 27, 27.

Grades obtained for Assignments 1+2+3+Mdtr+4:

173, 170, 166, 158, 156, 154, 154, 153, 153, 152, 136, 136, 135, 129, 127.

Assignment 5 due Wed Nov 14:

3.1.4, 3.2.4(i), 6.2.2 , 6.2.3, 6.2.9, 6.2.10
Grades obtained for Assignment 5:

36, 36, 35, 35, 35, 33, 32, 32, 32, 32, 32, 30, 30, 29, 29.

Grades obtained for Assignments 1+2+3+Mdtr+4+5:

206, 205, 198, 195, 189, 189, 188, 188, 183, 182, 168, 168, 166, 159, 158.

Assignment 6 due Wed Nov 28:

6.3.6, 6.3.12, 6.3.13, 6.7.4 (i) and (ii), 6.8.4 (ii), 6.8.5.

Hints to Chapter 6:

2.2 If ξ_t is right continuous, then $\xi_t = \lim \xi_{\kappa_n(t)}$, where $\kappa_n(t) = 2^{-n}[2^n t] + 2^{-n}$.
2.4 If $a = -1$ and $b = 1$, then for every $t \geq 0$

$$\{\omega : \tau > t\} = \{\omega : \sup_{r \in \rho \cup \{t\}} \xi_r^2 < 1\},$$

where ρ is the set of all rational numbers on $[0, \infty)$.
2.9 Define $\tau = \inf\{t \geq 0 : |\xi_t| \geq c\}$ and use Chebyshev's inequality.
2.10 In Theorem 2.8 and Exercise 2.9 put $N = c^2$ and integrate with respect to c over $(0, \infty)$.
3.6 Use Exercise 2.9.
3.11 Use Davis's inequality.
3.12 See the proof of Theorem 2.4.1 in order to get that $\tau < 1$ and $(1-s)^{-1} I_{s<\tau} \in \mathcal{S}$. Then prove that, for each $t \geq 0$, on the set $\{t \geq \tau\}$ we have (a.s.)

$$\int_0^t \frac{1}{1-s} I_{s<\tau}\, dw_s = \int_0^\tau \frac{1}{1-s}\, dw_s.$$

3.13 Use Davis's inequality.
4.3 Use Exercise 3.2.5 and Fatou's theorem for conditional expectations.
6.2 In (i) consider $\{\tau \leq t\}$.
6.5 Approximate stopping times with simple ones and use Bachelier's theorem.
7.4 In (iii) use the fact that

$$\int_0^n s^{-1} e^{-|x|^2/(2s)}\, ds - \int_0^n s^{-1} e^{-1/(2s)}\, ds = \int_n^{n|x|^{-2}} s^{-1} e^{-1/(2s)}\, ds \to -2 \ln |x|$$

as $n \to \infty$.
8.4 For appropriate stopping times $\tau_n \to \infty$, the processes $\rho_{t \wedge \tau_n}$ are martingales on $[0, T]$ and the processes $\rho^p_{t \wedge \tau_n}$ are submartingales. By Doob's inequality conclude that $E \sup_{t \leq T} \rho^p_{t \wedge \tau_n} \leq N$.

8.10 Observe that for $\mu = \lambda\tau^{1/2}$ and $r = w_\tau \tau^{-1/2}$ we have $\exp(\lambda w_\tau - \lambda^2\tau/2) = \exp(\mu r - \mu^2/2) =: f(r,\mu)$. Furthermore, Leibnitz's rule shows that $f_\mu^{(2k)}(r,0)$ is a polynomial (called a Hermit polynomial) in r of degree $2k$ with nonzero free coefficient.

10.3 In (i) take any smooth decreasing function $u(x)$ such that $u(x) > 0$ for $x < c$ and $u(x) = 0$ for $x \geq c$ and prove that $u(\xi_t) = \int_0^t u'(\xi_t)b(\xi_t)\,dt$. By comparing the signs of the sides of this formula conclude $u(\xi_t) = 0$.

10.4 Observe that $E\xi_{t\wedge\tau(r)} = E(t \wedge \tau(r))$.

Grades obtained for Assignment 6:

36, 34, 33, 33, 33, 32, 31, 31, 31, 30, 29, 28, 23, 19, 13.

Grades obtained for Assignments 1+2+3+Mdtr+4+5+6:

241, 236, 231, 223, 222, 220, 220, 220, 215, 212, 201, 190, 189, 187, 171.

Assignment 7 due Wed Dec 12:

6.8.6, 6.10.1, 6.10.3 in Part (ii) you additionally assume that $b \geq \delta$, where δ is a constant and $\delta > 0$, 6.10.4, 4.1.15, 4.2.8.

Hints to Chapter 4

1.11 Instead of Fourier integrals, consider Fourier series.

1.14 Use that continuous H-valued functions are uniformly continuous.

1.15 Observe that our assertions can be expressed in terms of R only, since, for every continuous nonrandom f,

$$E\Big|\int_a^b \xi_t f_t\,dt\Big|^2 = E\Big|\int_a^b \eta_t f_t\,dt\Big|^2$$

whenever ξ_t and η_t have the same correlation function. Another useful observation is that, if $R(0) = 1$, then $R(t) = Ee^{it\xi} = F\{0\} + Ee^{it\xi}I_{\xi\neq 0}$, and

$$\frac{1}{T}\int_0^T R(t)\,dt = F\{0\} + EI_{\xi\neq 0}[e^{iT\xi} - 1]/(iT\xi).$$

3.3 In the proof of the converse, notice that, if $R(0) = 1$, then ξ_r and $\xi_{t+s} - e^{-\alpha s}\xi_t$ are uncorrelated, hence independent for $r \leq t$, $s \geq 0$.

3.4 Write the left-hand side as the mean-square limit of integral sums, and use the isometric property of the stochastic integral along with the dominated convergence theorem to find the L_2-limit.

4.1 From $P_m(x)\tilde P_{\tilde n}(x) \equiv \tilde P_{\tilde m}(x)P_n(x)$ conclude that any root of P_m is a root of $\tilde P_{\tilde m}$, but not of P_n since P_m and P_n do not have common roots. Then derive that $\tilde P_{\tilde m}(x) \equiv P_m(x)$.

4.3 Observe that $\bar\varphi(x)|_{x=z} = \varphi(-x)|_{x=z}$ for all complex z and $\bar\varphi(x)|_{x=-iy} = \overline{\varphi(iy)}$ for real y.

5.1 Define

$$G(t) = \int_{\mathbb{R}} e^{itx} g(x)\frac{1}{Q_+(x)}\,dx$$

and prove that G is $m/2 - 1$ times continuously differentiable in t and tends to zero as $|t| \to \infty$ as the Fourier transform of an L_1 function. Then prove that G satisfies the equation $P_+(-iD_t)G(t) = 0$ for $t \leq 0$, where $D_t = d/dt$. Solutions of this linear equation are linear combinations of some integral powers of t times exponential functions. Owing to the choice of P_+, its roots lie in the closed upper half plane, which implies that the exponential functions are of type $\exp(at)$ with $\mathrm{Re}\, a \leq 0$, none of which goes to zero as $t \to -\infty$. Since $G(t) \to 0$ as $t \to -\infty$, we get that $G(t) = 0$ for $t \leq 0$. Now apply linear differential operators to G to get the conclusion.

5.2 Remember the definition of L_2^ξ from Remark 2.4. By this remark, if $\eta \in L_2^\xi$, then $\eta = \int_{\mathbb{R}} g(x)\, \lambda(dx)$ with $g \in L_2(\mathfrak{B}(\mathbb{R}), \ell)$. If in addition $\eta \perp L_2^\xi(-\infty, 0)$, then $\int_{\mathbb{R}} \bar{g}(x) e^{itx} \varphi(x)\, dx = 0$ for $t \leq 0$. Exercise 5.1 shows then that η is orthogonal to the random variables in (5.1).

5.8 For the uniqueness see the hint to Exercise 5.1. Also notice that P_+^ε does not have real roots, and

$$\tilde{\xi}_t^\varepsilon = \int_{\mathbb{R}} \frac{P_+(x)}{P_+^\varepsilon(x) Q_+(x)} e^{itx}\, \lambda(dx).$$

6.2 If the distributions of two vectors coincide, the distributions of their respective subvectors coincide too. Therefore, for any $i \leq n$, the vectors $(\xi_i, ..., \xi_n)$ and $(\xi_{i+1}, ..., \xi_{n+1})$ have the same distribution.

6.12 Notice that the process $\eta_n := z^n e^{i\omega}$, where ω is $\Omega = [0, 2\pi]$ with Lebesgue measure, is stationary. Also notice that the product of two independent stationary processes is stationary.

6.16 For an invariant set A and any integers $m \in \mathbb{R}$ and $k \geq 0$ we have

$$\int_\Omega e^{2\pi i m\omega} I_A(\omega)\, d\omega = e^{2\pi i m\alpha} \int_\Omega e^{2\pi i m\omega} I_A(\omega)\, d\omega$$
$$= e^{2\pi i mk\alpha} \int_\Omega e^{2\pi i m\omega} I_A(\omega)\, d\omega,$$

where $d\omega$ is the differential of the linear Lebesgue measure and k is any integer. By using (1.2.6), conclude that, for any square-integrable random variable f, $EfI_A = P(A)Ef$. Then take $f = I_A$.

Grades obtained for Assignment 7:

36, 36, 35, 34, 34, 34, 34, 34, 33, 30, 29, 27, 25, 21, 12.

Grades obtained for Assignments 1+2+3+Mdtr+4+5+6+7:

275, 272, 264, 256, 255, 254, 250, 249, 241, 237, 235, 224, 214, 210, 196.

Take home final due Dec 17 before 1pm:

3.3.13, 4.3.3, 4.4.3, 6.8.10

Grades obtained for the Final:

36, 36, 35, 35, 35, 35, 35, 34, 32, 29, 28, 26, 26, 24, 23.

Total grades:

310, 300, 298, 291, 291, 289, 285, 285, 272, 270, 267, 247, 240, 236, 220.

Gradelines:

$B = \{220\}, B+ = [236, 247], A- = [267, 272], A = [285, 310]$.