Solutions for HW 1 due Wed Sept 19:

1.1.7 (i) Notice that for any $B \subset \Omega$ we have $B = B \cup \emptyset$ with $P(\emptyset) = 0$. It follows that $\mathcal{F} \subset \mathcal{F}^P$. In particular, $\Omega \in \mathcal{F}^P$.

(ii) If $D_1, D_2, ... \in \mathcal{F}^P$, then, for some $C_n, B_n \in \mathcal{F}$ and $A_n \subset B_n$, we have $D_n = C_n \cup A_n$ and $P(B_n) = 0$. Then $\bigcup_n D_n = C \cup A$, where

$$C = \bigcup_n C_n \in \mathcal{F}, \quad A = \bigcup_n A_n \subset B := \bigcup B_n \in \mathcal{F},$$

and

$$P(B) \le \sum_n P(B_n) = 0.$$

Hence $\bigcup_n D_n \in \mathcal{F}^P$.

(iii) Notice that $A \in \mathcal{F}^P$ if A has zero probability ($A = \emptyset \cup A$, $A \subset B \in \mathcal{F}$, $P(B) = 0$). Also notice that any subset of a set of zero probability has zero probability.

Now, if A has zero probability: $A \subset B \in \mathcal{F}$ and $P(B) = 0$, then $B \setminus A \subset B$ and by the above $B \setminus A \in \mathcal{F}^P$. From $A^c = B^c \cup (B \setminus A)$ and (ii) we infer that $A^c \in \mathcal{F}^P$.

For general $C = D \cup A \in \mathcal{F}^P$ with $D \in \mathcal{F}$ and A being a set of probability zero, we have $C^c = D^c \cap A^c$, where $A^c \in \mathcal{F}^P$ that is $A^c = B \cup N$, with $B \in \mathcal{F}$ and N having zero probability. Then $C^c = (D^c \cap B) \cup (D^c \cap N)$, where $D^c \cap B \in \mathcal{F}$ and $D^c \cap N \subset N$ has zero probability. It follows by definition that $C^c \in \mathcal{F}^P$ and we are done.

1.1.9 We take for granted that
- f is a continuous nondecreasing function, $f(0) = 0, f(1) = 1$;
- If $x, y \in [0, 1]$, $x \ne y$, and $f(x) = f(y)$, then $f(x)$ is a rational number;
- If $x \in [0, 1]$ is irrational, then $f^{-1}(x) \in K$, where K is the Cantor set having Lebesgue measure zero.

First we prove

Lemma HW1.1. *For any Borel $B \subset [0, 1]$, $f(B)$ is a Borel subset of $[0, 1]$.*

Proof. As usual we denote by Σ the collection of all subsets of $[0, 1]$ whose image under mapping f is Borel. It turns out that Σ is a σ-field.

We start to check the definition of σ-field by noticing that $[0, 1] \in \Sigma$ since $f([0, 1]) = [0, 1]$. Furthermore, obviously, for any family (B_α) of subsets of $[0, 1]$,

$$f\big(\bigcup_\alpha B_\alpha\big) = \bigcup_\alpha f(B_\alpha).$$

Therefore, if $B_1, B_2, ... \in \Sigma$, then $\bigcup_n B_n \in \Sigma$. To finish proving that Σ is a σ-field it only remains to check that $B^c \in \Sigma$ if $B \in \Sigma$.

To do this step first observe that for any set $B \subset [0, 1]$

$$f(B^c) = \big([0, 1] \setminus f(B)\big) \cup \big(f(B) \cap f(B^c)\big). \tag{HW1.1}$$

Indeed, if $x \in f(B^c)$, then either $x \in f(B)$, in which case $x \in f(B) \cap f(B^c)$, or $x \notin f(B)$ that is $x \in [0, 1] \setminus f(B)$. In both cases x belongs to the right-hand side of (HW1.1). On the other hand, if x belongs to the right-hand side of (HW1.1), then either $x \in f(B) \cap f(B^c)$, in which case $x \in f(B^c)$, or else $x \notin f(B)$ but then, since x is the image of a point $y \in [0, 1]$, we have $y \notin B$, $y \in B^c$, that is $x \in f(B^c)$. In both cases $x \in f(B^c)$. This proves (HW1.1).

Next observe that, for any $B \subset [0,1]$, the set $f(B) \cap f(B^c)$ is countable, because for each $x \in f(B) \cap f(B^c)$ there are $y_1 \in B$ and $y_2 \in B^c$ such that $x = f(y_1) = f(y_2)$. Since $y_1 \neq y_2$, x is rational. The countable sets being Borel, (HW1.1) proves that $f(B^c)$ is Borel if $f(B)$ is Borel.

Thus, Σ is a σ-field indeed. Moreover Σ contains all closed subsets of $[0,1]$ since the continuous image of a compact set is compact. It follows that Σ contains the Borel σ-field on $[0,1]$, which is exactly what is asserted in the lemma.

Now, let C be a non Lebesgue measurable subset of $[0,1]$. We have to prove $f^{-1}(C)$ is not Borel. But if it were, due to $C = f(f^{-1}(C))$ and Lemma HW1.1 we would have that C is Borel.

Finally, assume C does not contain rational points. Then $f^{-1}(C) \subset K$, and, since K has Lebesgue measure zero, so does $f^{-1}(C)$.

2.6 (i)$\Longrightarrow$(ii). Let $a_1, a_2, ... \in K$ and, for each integer $n \geq 1$, let $\{x_1^n, ..., x_{k(n)}^n\}$ be a 2^{-n}-net for K. Then in 2^{-1}-neigborhood of at least one of $x_1^1, ..., x_{k(1)}^1$ there are infinitely many members of the sequence $a_1, a_2, ...$ Number them as $a_{n(1,1)}, a_{n(1,2)}, ...$ Notice that

$$n(1,i) \geq i, \quad |a_{n(1,i)} - a_{n(1,j)}| \leq 1 \quad \forall i,j.$$

Next in 2^{-2}-neigborhood of at least one of $\{x_1^2, ..., x_{k(2)}^2\}$ there are infinitely many members of the sequence $a_{n(1,1)}, a_{n(1,2)}, ...$ Number them as $a_{n(2,1)}, a_{n(2,2)}, ...$ Notice that

$$\{n(2,1), n(2,2), ...\} \subset \{n(1,1), n(1,2), ...\}, \quad n(2,i) \geq n(1,i), \quad |a_{n(2,i)} - a_{n(2,j)}| \leq 1/2 \quad \forall i,j.$$

By continuing in this way, for $k = 1, 2, ...$, we construct sequences $\{a_{n(k,1)}, a_{n(k,2)}, ...\}$ enjoying the following properties: $\{n(k+1,1), n(k+1,2), ...\} \subset \{n(k,1), n(k,2), ...\}$,

$$n(k,i) \geq i, \quad |a_{n(k,i)} - a_{n(k,j)}| \leq 2^{-k+1} \quad \forall i,j.$$

Then $|a_{n(k,k)} - a_{n(r,r)}| \leq 2^{-m+1}$ if $k, r \geq m$, so that $\{a_{n(k,k)}, k = 1, 2, ...\}$, is a Cauchy sequence and, since $n(k,k) \to \infty$, it is as subsequence of $\{a_1, a_2, ...\}$. The sequence $\{a_{n(k,k)}, k = 1, 2, ...\}$ converges to a point $a \in X$ (X is complete). Since K is closed, this point belongs to K.

(ii)$\Longrightarrow$(i). If for an $\varepsilon > 0$ there is no finite $\varepsilon > 0$-net for K, then take any $x_1 \in K$, notice that $\{x_1\}$ is not a ε-net for K so that there is a $x_2 \in K \setminus B_\varepsilon^o(x_1)$. Furthermore $\{x_1, x_2\}$ is not a ε-net for K so that there is a $x_3 \in K \setminus B_\varepsilon^o(x_1) \setminus B_\varepsilon^o(x_2)$. In this way one can find infinitely many different $x_1, x_2, ... \in K$, such that $|x_i - x_j| \geq \varepsilon$ for $i \neq j$. But then, obviously, this sequence does not have converging subsequences let alone converging to a point in K. One also easily proves the fact that K is closed.

1.2.13 One has (1.2.6) since α is irrational so that $m\alpha$ is not an integer and $\exp(i2\pi m\alpha) \neq 1$. Next, each continuous 1-periodic function f can be uniformly approximated by 1-periodic trigonometric polynomials. On 1-periodic trigonometric polynomials one has (1.2.7) due to (1.2.6) and the fact that (1.2.7) is obvious if f is constant. To finish proving (1.2.7) for continuous 1-periodic functions one observes that, for any 1-periodic trigonometric polynomial g,

$$\varlimsup_{n\to\infty} \Big| \int_0^1 f(y)\,dy - \tfrac{1}{n+1} \sum_{k=0}^{n} f(x+k\alpha) \Big| \leq I_1$$

$$+ \lim_{n\to\infty} \Big| \int_0^1 g(y)\,dy - \tfrac{1}{n+1} \sum_{k=0}^{n} g(x+k\alpha) \Big| + I_2 = I_1 + I_2,$$

where

$$I_1 := \Big| \int_0^1 f(y)\,dy - \int_0^1 g(y)\,dy \Big| \le \sup|f-g|,$$

$$I_2 := \overline{\lim_{n\to\infty}} \Big| \tfrac{1}{n+1} \sum_{k=0}^{n} f(x+k\alpha) - \tfrac{1}{n+1} \sum_{k=0}^{n} g(x+k\alpha) \Big| \le \sup|f-g|.$$

Let $\{y\}$ denote the fractional part of y. For 1-periodic functions, the sum in (1.2.7) is obviously the integral against the probability measure μ_n which charges each point $\{x+k\alpha\}$, $k=0,...,n$, with mass $1/(n+1)$. By the way, $\{x+k\alpha\} \ne \{x+j\alpha\}$ if $k \ne j$ since α is irrational. We claim that $\mu_n \overset{w}{\to} \ell$ that is

$$\int_0^1 f(y)\,\mu_n(dy) \to \int_0^1 f(y)\,dy \tag{HW1.2}$$

for all continuous functions f given on $[0,1]$. By the above we have this converegence if additionally $f(0)=f(1)$, in which case f is the restriction on $[0,1]$ of a 1-periodic continuous function.

There are few ways to do this. For instance, one can approximate any continuous function f given on $[0,1]$ with continuous functions g satisfying $g(0)=g(1)$ changing f only near the points 1 and 0. Then

$$\overline{\lim_{n\to\infty}} \Big| \int_0^1 f(y)\,\mu_n(dy) - \int_0^1 f(y)\,dy \Big| \le \overline{\lim_{n\to\infty}} \int_0^1 |f(y)-g(y)|\,\mu_n(dy) + \int_0^1 |f(y)-g(y)|\,dy. \tag{HW1.3}$$

Furthermore, one can arrange to have a continuous and 1-periodic function h such that $|f(y)-g(y)| \le h(y)$. Then, due to (HW1.2), the right-hand side of (HW1.3) is less than $\int_0^1 h\,dy$, which can be made arbitrary small.

Another a little bit more inspiring way to prove (HW1.2) starts by wrapping $[0,1]$ around a circle. In other words, take $g(y)=\exp(i2\pi y)$ and notice that g takes equal values at end points of the interval $[0,1]$ and maps it continuously onto the unit circle which we denote by A. For any continuous function h on A, upon substituting $f(y)=h(g(y))$ in (HW1.2) and using Theorem 1.1.13, we get

$$\int_A h(z)\,\mu_n g^{-1}(dz) \to \int_A h(z)\,\ell g^{-1}(dz). \tag{HW1.4}$$

Hence $\mu_n g^{-1} \overset{w}{\to} \ell g^{-1}$. As everybody knows, ℓg^{-1} is proportional to Lebesgue measure on the unit circle and therefore, due to Corollary 1.2.12, we have that (HW1.4) holds for any Borel Riemann integrable function h, that is, for any such function,

$$\int_0^1 h(g(y))\,\mu_n(dy) \to \int_0^1 h(g(y))h(y)\,dy. \tag{HW1.5}$$

Now, for any continuous f given on $[0,1]$, one can choose h discontinuous at most at only one point $z=1$ such that $h(g(y))=f(y)$ if $y\ne 1, y\ne 0$. Then one gets (HW1.2) from (HW1.5) since $\mu_n(\{0\})=\mu_n(\{1\})=0$.

We conclude doing this exercise by using Corollary 1.2.12 which shows that (HW1.2) implies that $\int_0^1 f\,\mu_n(dx) \to \int_0^1 f\,dx$ if f is the indicator of an interval.

1.2.14 By the provided hint $N_b(n)$ is the number of $i = 1, ..., n$ such that $k + \log_{10} b \le i \log_{10} 2 < k + \log_{10}(b+1)$ for some integer k, that is, in the notation from above, the number of $i = 1, ..., n$ such that

$$\log_{10} b \le \{i \log_{10} 2\} < \log_{10}(b+1).$$

Since $\log_{10} 2$ is irrational, the previous exercise yields that $N_b(n)/n \to \log_{10}(b+1) - \log_{10} b > 0$.

1.2.15 First notice that

$$\sup_{y:|y-x|<\varepsilon} f(y), \qquad \inf_{y:|y-x|<\varepsilon} f(y)$$

are monotone functions of ε. Therefore $\bar{f}$ and $\underline{f}$ from the hint are well defined.

If $\bar{f}(x) < c$, then there exist $\delta, \varepsilon > 0$ such that $\sup_{y:|y-x|<\varepsilon} f(y) < c - \delta$, that is in the ε-neigborhood of x we have $f(y) < c - \delta$. Then obviously $\bar{f}(y) \le c - \delta < c$ implying that the set $\{x : \bar{f} < c\}$ is open for any c and $\bar{f}$ is a Borel function. Similarly, $\underline{f}$ is Borel and $A := \{x : \bar{f}(x) \ne \underline{f}(x)\}$ is a Borel set.

Next, the inclusion $A \subset \Delta$ is obvious. To prove the opposite observe that if $x \in \Delta_f$, then either the limit of $f(y)$ as $y \to x$ does not exist, in which case

$$\bar{f}(x) \ge \overline{\lim_{y\to x}} f(y) > \underline{\lim}_{y\to x} f(y) \ge \underline{f}(x),$$

or the limit exists but is different from $f(x)$. Let g denote the limit. If $g < f(x)$, then $\bar{f}(x) = f(x) > g = \underline{f}(x)$. In the remaining case $g > f(x)$ we again have $\bar{f}(x) > \underline{f}(x)$. Thus $\Delta \subset A$.

Solutions for HW 2 due Wed Oct 3:

1.1.10 (ii) Denote $\Sigma = \{B : B \subset X, \xi^{-1}(B) \in \mathcal{F}\}$. First $\Omega = \xi^{-1}(X) \in \mathcal{F}$, therefore $X \in \Sigma$. Next by part (i), if $B \in \Sigma$, then $\xi^{-1}(B^c) = (\xi^{-1}(B))^c \in \mathcal{F}$ since $\xi^{-1}(B) \in \mathcal{F}$. Finally, again by part (i), if $B_1, B_2, ... \in \Sigma$, then $\xi^{-1}(\cup B_n) = \cup \xi^{-1}(B_n) \in \mathcal{F}$ since $\xi^{-1}(B_n) \in \mathcal{F}$. Hence $\cup B_n \in \Sigma$.

1.3.3 Denote by Σ the family in question. First, for any fixed $t \in T$, $X^T = \{x. : x_t \in X\} \in \Sigma$. Next if $B^{(n)} \in \mathfrak{B}^n$, $B^{(k)} \in \mathfrak{B}^k$, $t_1, ..., t_n, s_1, ..., s_k \in T$, $C_1 = \{x. : (x_{t_1}, ..., x_{t_n}) \in B^{(n)}\}$, $C_2 = \{x. : (x_{s_1}, ..., x_{s_k}) \in B^{(k)}\}$, then $B^{(n)} \times B^{(k)} \in \mathfrak{B}^{n+k}$ and

$$C_1 \cap C_2 = \{x. : (x_{t_1}, ..., x_{t_n}, x_{s_1}, ..., x_{s_k}) \in B^{(n)} \times B^{(k)}\} \in \Sigma.$$

Finally, $C_1^c = \{x. : (x_{t_1}, ..., x_{t_n}) \in X^n \setminus B^{(n)}\} \in \Sigma$.

1.3.5 Let X be the set consisting of only two numbers -1 and 1. Then any continuous X-valued function on $[0,1]$ is identically equal either to -1 or to 1. However there is no countably many points $t_1, t_2, ... \in [0,1]$ such that any function $x.$ on $[0,1]$ satisfying $x_{t_k}^2 = 1$ for all k also satisfies $x_t \equiv 1$ on $[0,1]$.

1.4.11 By formula (3), if $k \ge n$ and $|t - s| \le 2^{-k}$ and t, s are binary rational, then

$$|x_t - x_s| \le g(2^{-k}) + 2 \sum_{m=k+1}^{\infty} g(2^{-m}) \le 2 \sum_{m=k}^{\infty} g(2^{-m}).$$

Since g is increasing, we have

$$g(a) = g(a)(\ln(b/a))^{-1}\int_a^b y^{-1}\,dy \le (\ln(b/a))^{-1}\int_a^b y^{-1}g(y)\,dy \quad \forall b>a>0,$$

$$g(2^{-m}) \le (\ln 2)^{-1}\int_{2^{-m}}^{2^{-m+1}} y^{-1}g(y)\,dy.$$

Thus, $|x_t - x_s| \le NG(2^{-k+1})$ if $k \ge n$, $|t-s| \le 2^{-k}$, and t,s are binary rational. By taking $k = [-\log_2|t-s|]$ for $|t-s| \le 2^{-n}$ we get

$$|t-s| \le 2^{-k}, \quad 2^{-k+1} \le 4\cdot 2^{\log_2|t-s|} = 4|t-s|, \quad |x_t - x_s| \le NG(4|t-s|).$$

1.4.13 We have

$$P(\Omega_n^c) \le \sum_{m=n}^{\infty}\sum_{i=0}^{2^m-1} P\big(|\xi_{(i+1)/2^m} - \xi_{i/2^m}| > ag(2^{-m})\big),$$

which by Exercise 12 is less than

$$I := (2/\sqrt{2\pi})\sum_{m=n}^{\infty} R^{1/2}(2^{-m})a^{-1}g^{-1}(2^{-m})2^m \exp\big(-a^2g^2(2^{-m})2^{-1}R^{-1}(2^{-m})\big).$$

We transform the last expression by using the definition $g(x) = \sqrt{R(x)(-\ln x)}$ which means that $g(2^{-m})R^{-1/2}(2^{-m}) = \sqrt{m\ln 2}$. Then we find

$$I = (2/\sqrt{2\pi})(\ln 2)^{-1/2}a^{-1}\sum_{m=n}^{\infty} 2^m m^{-1/2}\exp(-a^2 2^{-1} m\ln 2) = Na^{-1}\sum_{m=n}^{\infty}\frac{1}{\sqrt{m}}2^{m(1-a^2/2)}.$$

If $a > \sqrt{2}$, the last series converges. Therefore, $P(\Omega') = 1$ that is with probability one there is an integer n such that for all $m \ge n$ and $i = 0, ..., 2^m - 1$ we have

$$|\xi_{(i+1)/2^m} - \xi_{i/2^m}| \le ag(2^{-m}). \tag{HW2.1}$$

By Exercise 11 for almost any ω the trajectory $\xi_\cdot(\omega)$ is uniformly continuous on binary rational numbers of $[0,1]$. Then as in the proof of Theorem 8 we conclude that ξ_t indeed has a continuous modification. At this moment we notice that

$$\lim_{s\to t} E|\xi_t - \xi_s|^2 \le \lim_{s\to t} R(|t-s|) = 0$$

since, if the last limit were > 0, the integral defining G would diverge.

To prove the last assertion in the exercise, it suffices to notice that $g(x) = N|\ln x|^{1/2-p/2}$ is an increasing function on $[0,1]$ and

$$G(x) = \int_0^x y^{-1}g(y)\,dy = N\int_{-\ln x}^{\infty} z^{1/2-p/2}\,dz < \infty$$

if $p > 3$.

1.4.13 Fix $a > \sqrt{2}$. By virtue of (HW2.1) and Exercise 11 we have $|\tilde\xi_t - \tilde\xi_s| \le NG(4|t-s|)$ if t,s are binary rational and $|t-s| \le 2^{-n(\omega)}$, where $n(\omega) < \infty$ with probability 1. Since $\tilde\xi_\cdot$ is a continuous function, the inequality $|\tilde\xi_t - \tilde\xi_s| \le NG(4|t-s|)$ holds for all $t,s \in [0,1]$ such that $|t-s| \le 2^{-n(\omega)}$.

Solutions for HW 3 due Wed Oct 17:

2.2.5 Since $Ew_t = 0$ and $E|w_t - w_s|^2 = |t-s|$, we can take $R(x) = x$ in Exercise 1.4.13. Then $g(x) = \sqrt{x|\ln x|}$ and, for $x \in [0, e^{-1}]$, (change of variables $y \to e^{-y}$)

$$G(x) = \int_0^x y^{-1} g(y)\,dy = \int_{-\ln x}^{\infty} e^{-y/2}\sqrt{y}\,dy = -2e^{-y/2}\sqrt{y}\big|_{-\ln x}^{\infty} + \int_{-\ln x}^{\infty} e^{-y/2} y^{-1/2}\,dy$$

$$\le 2\sqrt{x|\ln x|} + \int_{-\ln x}^{\infty} e^{-y/2}\,dy \le 4\sqrt{x|\ln x|}.$$

It only remains to notice that by Exercise 1.4.13 there is a constant N such that for almost any ω one can find an integer $n \ge 3$ such that $|w_t - w_s| \le NG(4|t-s|)$ whenever $t, s \in [0,1]$ and $|t-s| \le 2^{-n}$.

2.2.10 First notice

$$P\{\max_{s\le 1} w_s \ge b, w_1 \le a\} = P\{\max_{s\le 1} w_s \ge b\} - P\{\max_{s\le 1} w_s \ge b, w_1 \ge a\}$$

and by Bachelier's theorem

$$P\{\max_{s\le 1} w_s \ge b\} = P(|w_1| \ge b) = \frac{2}{\sqrt{2\pi}}\int_b^{\infty} e^{-x^2/2}\,dx.$$

Next, if $b \le 0$, then since obviously $\max_{s\le 1} w_s \ge w_0 = 0$ (a.s.), we have

$$P\{\max_{s\le 1} w_s \ge b, w_1 \ge a\} = P\{w_1 \ge a\} = \frac{1}{\sqrt{2\pi}}\int_a^{\infty} e^{-x^2/2}\,dx. \tag{HW3.1}$$

Obviously (HW3.1) also holds if $a \ge b$.

Thus the only remaining case is $b > 0$ and $a < b$. We go back to the proof of Theorem 2.2.3 (Bachelier's theorem) slightly changing the notation and considering 2^{2n} in place on n. So let $m(n) = 2^{2n}$

$$\eta_t^n = \xi_t^{m(n)}, \quad \zeta^n = \max_{[0,1]} \eta_t^n$$

Similarly to our argument concerning formula (2.2.1) in the proof of Theorem 2.2.3, we get that, for any integers $j < i$ and $i > 0$,

$$P\{\max_{k\le m(n)} S_k \ge i, S_{m(n)} < j\} = P\{\max_{k\le m(n)} S_k \ge i, S_{m(n)} > 2i - j\}$$

and since $2i - j > i$,

$$P\{\max_{k\le m(n)} S_k \ge i, S_{m(n)} < j\} = P\{S_{m(n)} > 2i - j\}.$$

Hence (notice $\sqrt{m(n)} = 2^{-n}$)

$$P(\zeta^n \ge i2^{-n}, \xi_1^n < j2^{-n}) = P(\xi_1^n > 2i2^{-n} - j2^{-n}).$$

It follows that for all binary rational a and b satisfying $b > 0$ and $a < b$ and all large n (so that $a = j2^{-n}$ and $b = i2^{-n}$ for some integers i and j) we have

$$P(\zeta^n \ge b, \xi_1^n < a) = P(\xi_1^n > 2b - a). \tag{HW3.2}$$

Now we let $n \to \infty$ and use Donsker's theorem (Theorem 2.1.3) which says that the distributions of $\xi^n_\cdot$ converge to the distribution of $w_\cdot$. We also use the fact that the function

$$x_\cdot \to (\max_{[0,1]} x_t, x_1)$$

is continuous, so that by a vector-valued version of Exercise 1.2.9 the distribution of

$$(\max_{[0,1]} \xi^n_t, \xi^n_1)$$

converges to that of $(\max_{[0,1]} w_t, w_1)$. Finally, we use the portmanteau theorem (Theorem 1.2.11 (iv)) and conclude from (HW3.2) that

$$P(\max_{[0,1]} w_t \geq b, w_1 < a) = P(w_1 > 2b - a) = \frac{1}{\sqrt{2\pi}} \int_{2b-a}^{\infty} e^{-x^2/2}\, dx \tag{HW3.3}$$

if a and b are binary rational such that $b > 0$ and $a < b$ and

$$P(\max_{[0,1]} w_t = b) = P(w_1 = a) = P(w_1 = 2b - a) = 0$$

The latter condition is always satisfied since $\max_{[0,1]} w_t$ and w_1 have densities. By the same reasons the expressions in (HW3.3) as functions of a or of b do not have jumps. Hence they are continuous in a and in b and from the fact that they coincide for all binary rational a and b such that $b > 0$ and $a < b$ it follows that, actually, they coincide for all a and b such that $b > 0$ and $a < b$. By combining the above results we obtain what we were after.

2.2.11 Since $w_t \sim N(0, t)$ we have $Ee^{\lambda w_t} = \exp(\lambda^2 t/2)$ for all complex λ. This implies

$$Q(\Omega) = \int_\Omega e^{w_1(\omega) - 1/2}\, P(d\omega) = Ee^{w_1 - 1/2} = 1.$$

Hence $(\Omega, \mathcal{F}, Q)$ is a probability space.

Define $\xi_t = w_t - t$, take $0 = t_0 \leq t_1 \leq \ldots \leq t_n = 1$ and compute the characteristic function of the vector $(\xi_{t_1}, \xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}})$ as a random variable on the probability space $(\Omega, \mathcal{F}, Q)$. This characteristic function is given by

$$\int_\Omega \exp i \sum_{k=1}^n \lambda_k (\xi_{t_k} - \xi_{t_{k-1}})\, Q(d\omega) = \int_\Omega e^{w_1 - 1/2} \exp i \sum_{k=1}^n \lambda_k (\xi_{t_k} - \xi_{t_{k-1}})\, P(d\omega)$$

$$= e^{-1/2 - i \sum_{k=1}^n \lambda_k (t_k - t_{k-1})} E e^{w_1} \exp i \sum_{k=1}^n \lambda_k (w_{t_k} - w_{t_{k-1}})$$

$$= e^{-1/2 - i \sum_{k=1}^n \lambda_k (t_k - t_{k-1})} E \exp \sum_{k=1}^n (1 + i\lambda_k)(w_{t_k} - w_{t_{k-1}})$$

$$= \exp\big(-1/2 - i \sum_{k=1}^n \lambda_k (t_k - t_{k-1}) + (1/2) \sum_{k=1}^n (1 + i\lambda_k)^2 (t_k - t_{k-1})\big)$$

$$= \exp\big(-(1/2) \sum_{k=1}^n \lambda_k^2 (t_k - t_{k-1})\big) = \prod_{k=1}^n e^{-(1/2)\lambda_k^2 (t_k - t_{k-1})}.$$

This shows that, on the probability space $(\Omega, \mathcal{F}, Q)$, the random variables $\xi_{t_1}, \xi_{t_2} - \xi_{t_1}, \ldots, \xi_{t_n} - \xi_{t_{n-1}}$ are independent and $\xi_{t_k} - \xi_{t_{k-1}} \sim N(0, t_k - t_{k-1})$.

2.3.8 By the law of large numbers $\sigma_n/n \to 1$ and $\sigma_n \to \infty$ (a.s.). Therefore π_t is finite, right continuous, and increasing thus has locally bounded variation. Looking at your homeworks I noticed that not everybody knows well the theory of integration against such functions. Therefore, below I dwell on some elements of this theory.

Being finite and increasing π_t defines a unique measure π on Borel σ-field of $(0,\infty)$ satisfying $\pi((s,t]) = \pi_t - \pi_s$ for all $0 \le s \le t < \infty$. Of course, by $\int_0^\infty f(t)\,d\pi_t$ we mean Lebesgue integral $\int_{(0,\infty)} f(t)\,\pi(dt)$, which is defined for any bounded Borel f with compact support (that is vanishing outside a finite interval). If $f = I_{(s,t]}$, then by definition

$$\int_0^\infty I_{(s,t]}(r)\,d\pi_r = \int_{(0,\infty)} I_{(s,t]}(r)\,\pi(dr) = \pi_t - \pi_s = \sum_{n=1}^\infty I_{\sigma\le t} - \sum_{n=1}^\infty I_{\sigma \le s}$$

$$= \sum_{n=1}^\infty I_{s<\sigma\le t} = \sum_{n=1}^\infty I_{(s,t]}(\sigma_n).$$

In short

$$\int_0^\infty I_B(r)\,d\pi_r = \sum_{n=1}^\infty I_B(\sigma_n) \qquad \text{(HW3.4)}$$

for all B from a π-system generating $\mathfrak{B}((0,\infty)$. By standard application of λ- and π-systems one concludes that (HW3.4) holds for all Borel B. By approximating uniformly and Borel bounded f with compact support by finite linear combinations of the indicators of Borel sets and passing to the limit we get that

$$\int_0^\infty f(r)\,d\pi_r = \sum_{n=1}^\infty f(\sigma_n)$$

for any bounded Borel f with compact support.

For each $k \ge 2$, the function

$$F(t_1,...,t_k) := \exp\big(i\sum_{n=1}^k f(s+t_1+...+t_n)\big)$$

is measurable. Since the joint distribution of $(\tau_1,...,\tau_k)$ is the product of the distributions of τ_i and τ_i's are iid, by using Fubini's theorem we easily get

$$E\exp\big(i\sum_{n=1}^k f(s+\sigma_n)\big) = EF(\tau_1,...,\tau_k) = \int_0^\infty e^{-t}\Phi(t)\,dt,$$

where

$$\Phi(t) = EF(t,\tau_1,...,\tau_{k-1}) = e^{if(s+t)}E\exp\big(i\sum_{n=1}^{k-1} f(s+t+\sigma_n)\big).$$

By letting $k\to\infty$ and observing that $\sigma_n \to \infty$ (a.s.) (by the law of large numbers), so that $f(s+\sigma_n) = 0$ for all n large enough, and by using the dominated convergence theorem we conclude

$$\varphi(s) = \int_0^\infty e^{if(s+t)-t}\varphi(s+t)\,dt, \quad (\varphi(s)e^{-s}) = \int_s^\infty e^{if(t)}(e^{-t}\varphi(t))\,dt.$$

It follows that $\varphi(s)e^{-s}$ is differentiable, in particular, continuous along with $e^{if(t)}(e^{-t}\varphi(t))$ (remember that Lebesgue integral is an absolutely continuous and hence continuous function of its limits). Also,

$$(\varphi(s)e^{-s})' = -e^{if(s)}(e^{-s}\varphi(s)), \quad \varphi(s)e^{-s} = C\exp\big(-\int_0^s e^{if(t)}\,dt\big),$$

where C is a constant. Next, $e^{if(t)}-1$ vanishes at infinity, since the support of f is bounded. Therefore,

$$\varphi(s) = C\exp\big(-\int_0^s (e^{if(t)}-1)\,dt\big) = C_1\exp\big(\int_s^\infty (e^{if(t)}-1)\,dt\big),$$

where C_1 is another constant. By using the dominated convergence theorem and the fact that $\sum_{n=1}^\infty f(s+\sigma_n)\to 0$ as $s\to\infty$ (a.s.), we conclude that $\varphi(s)\to 1$ and $\int_s^\infty(e^{if(t)}-1)\,dt\to 0$ as $s\to\infty$. It follows that $C_1=1$. Thus,

$$E\exp\{i\int_0^\infty f(t)\,d\pi_t\} = \exp(\int_0^\infty (e^{if(t)}-1)\,dt) \tag{HW3.5}$$

for any continuous f which vanishes outside a finite interval.

Now take an integer $n\geq 1$, $0=t_0,<t_1<\ldots<t_n$, and some real numbers $\lambda_1,\ldots,\lambda_n$. Introduce a function f by $f(t)=\lambda_k$ if $t\in(t_{k-1},t_k]$ and $f(t)\equiv 0$ for $t>t_n$. Then

$$\int_0^\infty f(t)\,d\pi_t = \sum_{k=1}^n \lambda_k(\pi_{t_k}-\pi_{t_{k-1}}).$$

Furthermore, for each k, it is easy to construct a sequence of uniformly bounded continuous functions vanishing for $t>t_n$ which converges at each point to $I_{(t_{k-1},t_k]}$. Their linear combinations will converge to f. Then by the dominated convergence theorem we find that (HW3.5) holds for our discontinuous f. Since $e^{if(t)}-1=\sum_{k=1}^n I_{(t_{k-1},t_k]}(e^{i\lambda_k}-1)$, this means

$$E\sum_{k=1}^n \lambda_k(\pi_{t_k}-\pi_{t_{k-1}}) = \exp\big(\sum_{k=1}^n (e^{i\lambda_k}-1)(t_k-t_{k-1})\big) = \prod_{k=1}^n \exp\big((e^{i\lambda_k}-1)(t_k-t_{k-1})\big).$$

All the remaning assertions of the exercise follow immediately from this formula.

2.3.14 Fix $T\in[0,\infty)$ and let Σ be the collection of all Borel susets B of $(0,T]$ such that

$$E\int_0^\infty I_B(t)\,d\pi_t = \ell(B).$$

By Exercise 3.8 we have that Σ contains any interval $(0,t]\in(0,T]$. Also almost obviously Σ is a λ-system. Therefore, by Lemma 2.3.18 the Borel σ-field of $(0,T]$ coincides with Σ. Then of course for any Borel step function f with compact support

$$E\int_0^\infty f(t)\,d\pi_t = \int_0^\infty f(t)\,dt. \tag{HW3.6}$$

By bearing in mind the monotone convergence theorem and remembering that any nonnegative Borel function is the monotone limit of step functions, we conclude that (HW3.6) holds for any nonnegative Borel function f with compact support.

Next, if $0 = t_0 < t_1 < ... < t_n \le 1$ and

$$f = \sum_{k=1}^{n} f(t_k) I_{[t_{k-1},t_k)}, \tag{HW3.7}$$

then

$$f = \sum_{k=1}^{n} f(t_k) I_{(t_{k-1},t_k]} \quad \text{(a.e.)}$$

and (cf. (2.3.7))

$$\text{(stoch)} \int_0^1 f(t)\, d(\pi_t - t) = \sum_{k=1}^{n} f(t_k)[(\pi_{t_k} - t_k) - (\pi_{t_{k-1}} - t_{k-1})]$$

$$= \text{(usual)} \int_0^1 f(t)\, d\pi_t - \int_0^1 f(t)\, dt \quad \text{(a.s.)} \tag{HW3.8}$$

The set of functions of type (HW3.7) is everywhere dense in $L_2(0,1)$. Therefore if we take a Borel $f \in L_2(0,1)$, we can find functions f_n of type (HW3.7) converging to f in $L_2(0,1)$. In that case

$$\text{(stoch)} \int_0^1 f(t)\, d(\pi_t - t) = \underset{n\to\infty}{\text{l.i.m.}} \text{(stoch)} \int_0^1 f_n(t)\, d(\pi_t - t),$$

$$E\Big|\text{(usual)} \int_0^1 f(t)\, d\pi_t - \text{(usual)} \int_0^1 f_n(t)\, d\pi_t\Big|$$

$$\le E\text{(usual)} \int_0^1 |f(t) - f_n(t)|\, d\pi_t = \int_0^1 |f(t) - f_n(t)|\, dt \le ||f - f_n||_2 \to 0$$

as $n \to \infty$. Thus, by passing to the limit in (HW3.8) we get what we wanted.

2.3.21 We have

$$E\exp(-\sum_n |\zeta((a_{n+1}, a_n])|) = E\exp(-\sum_n |w_{a_n} - w_{a_{n+1}}|) = \prod_n b_n,$$

where

$$b_n = E\exp(-|w_{1/(n+1)} - w_{1/n}|) = E\exp(-|w_1|/\sqrt{n(n+1)}\ )$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x/\sqrt{n(n+1)} - x^2/2}\, dx,$$

$$1 - b_n = \frac{2}{\sqrt{2\pi}} \int_0^\infty (1 - e^{-x/\sqrt{n(n+1)}}) e^{-x^2/2}\, dx$$

$$\sum_n (1 - b_n) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \sum_n (1 - e^{-x/\sqrt{n(n+1)}}) e^{-x^2/2}\, dx.$$

Upon noticing that $1 - e^{-\alpha} \sim \alpha$ as $\alpha \to 0$ and that $\sum_n x/\sqrt{n(n+1)} = \infty$ for any $x > 0$ we conclude

$$\sum_n (1 - e^{-x/\sqrt{n(n+1)}}) = \infty \quad \text{if} \quad x > 0, \quad \sum_n (1 - b_n) = \infty, \quad \prod_n b_n = 0.$$

Solutions for Midterm, Oct 22, 2001:

1. Denote

$$\xi_n = \sum_{i=0}^{2^n-1} (w_{(i+1)2^{-n}} - w_{i2^{-n}})^2.$$

Then

$$E\xi_n = \sum_{i=0}^{2^n-1} E(w_{(i+1)2^{-n}} - w_{i2^{-n}})^2 = \sum_{i=0}^{2^n-1} 2^{-n} = 1,$$

$$\operatorname{Var}\xi_n = \sum_{i=0}^{2^n-1} \operatorname{Var}\big((w_{(i+1)2^{-n}} - w_{i2^{-n}})^2\big) = \sum_{i=0}^{2^n-1} \operatorname{Var}\big((2^{-n/2}w_1)^2\big)$$

$$= \operatorname{Var}\big(w_1^2\big) \sum_{i=0}^{2^n-1} 2^{-2n} = 2^{-n}\operatorname{Var}\big(w_1^2\big).$$

Hence

$$\sum_{n=1}^{\infty} E(\xi_n - 1)^2 < \infty, \quad E\sum_{n=1}^{\infty}(\xi_n - 1)^2 < \infty, \quad \sum_{n=1}^{\infty}(\xi_n - 1)^2 < \infty \quad \text{(a.s.)},$$

and $(\xi_n - 1)^2 \to 0$ (a.s.).

2. If $f \in S(\Pi)$, then

$$\int_X f(x)\,\tilde{\zeta}_1(dx) = \int_X f(x)\,\zeta(dx) = \int_X f(x)\,\tilde{\zeta}_2(dx) \quad \text{(a.s.)} \tag{MT.1}$$

since $\tilde{\zeta}_i = \zeta$ on Π_0 and $\Pi_0 = \Pi$. By Theorem 2.3.13 equality (MT.1) holds for all $f \in L_2(\Pi, \mu)$. By Theorem 2.3.19 we have $L_2(\Pi, \mu) = L_2(\sigma(\Pi), \mu)$. Hence (MT.1) holds for all $f \in L_2(\sigma(\Pi), \mu)$ indeed. By taking $f = I_\Gamma$, we conclude that, for any $\Gamma \in \sigma(\Pi)$, $\tilde{\zeta}_1(\Gamma) = \tilde{\zeta}_2(\Gamma)$ (a.s.).

3. For any $a_1, ..., a_n \in (0, \infty)$ and $t_1, ..., t_n \in [0, \infty)$, by the self similarity of the Wiener process, we have

$$P(c\tau_{a_1/\sqrt{c}} > t_1, ..., c\tau_{a_n/\sqrt{c}} > t_n) = P(\tau_{a_1/\sqrt{c}} > t_1/c, ..., \tau_{a_n/\sqrt{c}} > t_n)$$

$$= P(\max_{s \le t_1/c} w_s < a_1/\sqrt{c}, ..., \max_{s \le t_n/c} w_s < a_n/\sqrt{c})$$

$$= P(\max_{s \le t_1} \sqrt{c}w_{s/c} < a_1, ..., \max_{s \le t_n} \sqrt{c}w_{s/c} < a_n) = P(\max_{s \le t_1} w_s < a_1, ..., \max_{s \le t_n} \sqrt{w}_s < a_n)$$

$$= P(\tau_{a_1} > t_1, ..., \tau_{a_n} > t_n).$$

Thus

$$\mu(B^{(n)}) := P\big((c\tau_{a_1/\sqrt{c}}, ..., c\tau_{a_n/\sqrt{c}}) \in B^{(n)}\big) = P\big((\tau_{a_1}, ..., \tau_{a_n}) \in B^{(n)}\big) = \nu(B^{(n)})$$

for any $B^{(n)} \in [0, \infty)^n$ of type

$$\{x \in [0, \infty)^n, x_1 > t_1, ..., x_n > t_n\} \tag{MT.2}$$

The collection of sets of type (MT.2) is a π-system and it is easy to see that the smallest σ-field containing all sets (MT.2) is the Borel σ-field of $[0, \infty)^n$. In addition, the collection of all sets $B^{(n)}$ on which μ and ν coinside contains $[0,\infty)^n$ (since $\tau_a \geq 0$) and is a λ-system. By Lemma 2.3.18 we conclude that (MT.2) holds for all Borel $B^{(n)}$ and we are done.

4. Since $w_{n+1} - w_n$ is independent of $(w_1, ..., w_n)$ and w_n is $\sigma(w_1, ..., w_n)$-measurable, we have (a.s.)

$$E(w_{n+1}|w_1, ..., w_n) = E(w_{n+1} - w_n + w_n|w_1, ..., w_n)$$
$$= E(w_{n+1} - w_n|w_1, ..., w_n) + E(w_n|w_1, ..., w_n) = E(w_{n+1} - w_n) + w_n = w_n,$$
$$E(e^{w_{n+1}-(n+1)/2}|w_1, ..., w_n) = E(e^{w_{n+1}-w_n-1/2}e^{w_n-n/2}|w_1, ..., w_n)$$
$$= e^{w_n-n/2}Ee^{w_{n+1}-w_n-1/2} = e^{w_n-n/2}.$$

Solutions for HW4 due Wed Oct 31:

2.3.23 For $t \in [0,1]$ denote

$$\xi_t = \int_0^t f(s)\,dw_s \quad \Big(:= \int_0^1 I_{s\leq t} f(s)\,dw_s \Big).$$

By Remark 2.3.15 we have $E\xi_t = 0$ and by formula (2.3.6) we have

$$E\xi_t\xi_s = E\Big(\int_0^1 I_{r\leq t} f(r)\,dw_r\Big)\Big(\int_0^1 I_{r\leq s} f(r)\,dw_r\Big) = \int_0^1 I_{r\leq t} f(r) I_{r\leq s} f(r)\,dr$$
$$= \int_0^{t\wedge s} f^2(r)\,dr = (\int_0^s f^2(u)\,du) \wedge (\int_0^t f^2(u)\,du).$$

Next we claim that ξ_1 is Gaussian. This is obvious if $f \in S(\Pi)$ with $\Pi = \{[0,t] : t \leq 1\}$. Indeed, if $f(s) = \sum_{i\leq n} c_i I_{[0,t_i]}(s)$ with $t_i \leq 1$, then $\xi_1 = \sum_{i\leq n} c_i w_{t_i}$ is Gaussian as a linear function of the Gaussian vector $(w_{t_1}, ..., w_{t_n})$.

For arbitrary $f \in L_2(\Pi)$ we can take a defining sequence $f_n \in S(\Pi)$ and remember that the limit in the mean of Gaussian variables is Gaussian. Then we see that our claim is true.

Finally, notice that the vector $(\xi_{t_1}, ..., \xi_{t_1})$ is Gaussian if and only if for any constants $\lambda_1, ..., \lambda_n$ the random variable $\lambda_1\xi_{t_1} + ... + \lambda_n\xi_{t_n}$ is Gaussian. Then it only remains to observe that

$$\lambda_1\xi_{t_1} + ... + \lambda_n\xi_{t_n} = \int_0^1 (\lambda_1 I_{s\leq t_1} + ... + \lambda_n I_{s\leq t_1}) f(s)\,dw_s,$$

where the last expression is Gaussian by the above argument.

Note. It is useful to remember that there exist Gaussian random variables ξ and η such that $\xi + \eta$ is <u>not</u> Gaussian.

2.5.9 Denote $u(x) = (b - x)(a - x)$. Then $x \in (a, b)$ if and only if $u(x) < 0$ and

$$\{\omega : \tau > t\} = \{\omega : \max_{s\leq t} u(w_s) < 0\}.$$

One can replace max with sup over rational numbers in $[0, t]$ and then we see that $u(w_s)$ and $\max_{s\leq t} u(w_s)$ are $\mathcal{F}_t$-measurable.

3.1.3 Let Σ be the collection of all unions of $A_{n(1)}, A_{n(2)}, ...$ for all possible sequences of distinct $n(1), n(2), ... \in \{1, 2, ...\}$. Obviously, due to $\Omega = \cup_n A_n$, Σ is a σ-field, and $\Sigma = \mathcal{G}$. Now we claim that any $\mathcal{G}$-measurable function η is constant on each A_n. Indeed, $\eta_n := 2^{-n}[2^n\xi]$, $n = 1, 2, ...$, are $\mathcal{G}$-measurable and take only countably many values of type $i2^{-n}$, $i = 0, \pm1, ...$ Therefore, the set $\{\omega : \eta_n(\omega) = i2^{-n}\}$ is the union of some A_k. This implies that η_n is constant on each A_k (either 0 or $i2^{-n}$). Let $\eta_n = a_{kn}$ on A_k. Then due to $|\eta - \eta_n| \le 2^{-n}$, on A_k we have $\eta = \lim_n \eta_n = \lim_n a_{kn}$, which is a constant. This proves our claim.

Finally, $\eta := E(\xi|\mathcal{G})$ exists and equals a constant, say a_k on A_k. By definition

$$a_k P(A_k) = E\eta I_{A_k} = E\xi I_{A_k}.$$

Hence, if $P(A_k) \neq 0$, we have $\eta = a_k = [P(A_k)]^{-1} E\xi I_{A_k}$ for all $\omega \in A_k$. Finally if $P(A_k) = 0$, then η equals any constant, say 0 almost surely on A_k since $P(\eta \neq 0, A_k) \le P(A_k) = 0$.

3.1.17 Denote π_L the orthogonal projection of $L_2(\mathcal{F}, P)$ on L. Use the property which characterizes uniquely the orthogonal projection saying that $\pi_L \xi \in L$ and $E(\xi - \pi_L \xi)^2$ is the least number among $E(\xi - \zeta)^2$, $\zeta \in L$. Also remember that by Theorem 3.1.14 (ii) we have $\eta := E(\xi|\xi_1, ..., \xi_n) \in L$. Then

$$E(\xi - \pi_L \xi)^2 \le E(\xi - \eta)^2. \tag{HW4.1}$$

On the other hand elements of L are $\sigma(\xi_1, ..., \xi_n)$-measurable and by Theorem 3.1.14 (i) we have the opposite inequality in (HW4.1). Thus, we have an equality in (HW4.1) and $E(\xi - \eta)^2$ is the least number among $E(\xi - \zeta)^2$, $\zeta \in L$. Since this property characterizes the projection, we conclude $\pi_L \xi$ and η coincide as elements of $L_2(\mathcal{F}, P)$.

3.2.3 The sufficiency is seen from the following

$$E(\xi_{n+1}|\mathcal{F}_n) = E\big(E(\eta_{n+1}|\mathcal{F}_{n+1})|\mathcal{F}_n\big) = E(\eta_{n+1}|\mathcal{F}_n) \ge E(\eta_n|\mathcal{F}_n) = \xi_n \quad \text{(a.s.)}.$$

The idea to prove necessity is best seen when $N = 2$, Then we only have to define $\eta_1 \le \eta_2 = \xi_2$, so that $\xi_1 = E(\eta_1|\mathcal{F}_1)$. But, basically,

$$\xi_1 = E(\xi_2|\mathcal{F}_1)\,\frac{\xi_1}{E(\xi_2|\mathcal{F}_1)} = E\left(\xi_2\,\frac{\xi_1}{E(\xi_2|\mathcal{F}_1)}\middle|\mathcal{F}_1\right), \quad \frac{\xi_1}{E(\xi_2|\mathcal{F}_1)} \le 1, \quad \xi_2\,\frac{\xi_1}{E(\xi_2|\mathcal{F}_1)} \le \eta_2.$$

Therefore, define

$$\zeta_n = \xi_n\big(E(\xi_{n+1}|\mathcal{F}_n)\big)^{-1} \quad n < N \quad (0 \cdot 0^{-1} := 0), \quad \zeta_N = 1,$$

$$\eta_n = \zeta_n \cdot ... \cdot \zeta_N \xi_N.$$

Then (remember that $0 \le \xi_n \le E(\xi_{n+1}|\mathcal{F}_k)$ almost surely) it holds that $0 \le \zeta_n \le 1$ for $n \le N$ and η_n increases with n (a.s.). Furthermore, $\eta_N = \xi_N$. Changing η_n on a set of probability zero, we can have η_n increasing and $\eta_N = \xi_N$ for all ω.

Next, for $n < N$, we have $\xi_n = \zeta_n E(\xi_{n+1}|\mathcal{F}_n)$ by definition on the set where $E(\xi_{n+1}|\mathcal{F}_n) \neq 0$ and almost surely on the set where $E(\xi_{n+1}|\mathcal{F}_n) = 0$ since $0 \le \xi_n \le E(\xi_{n+1}|\mathcal{F}_k)$ almost surely. By niticing that ζ_n are $\mathcal{F}_n$-measurable we conclude (a.s.)

$$\xi_n = \zeta_n E(\xi_{n+1}|\mathcal{F}_n) = E(\zeta_n \xi_{n+1}|\mathcal{F}_n).$$

Finally, we iterate this relation and get

$$\begin{aligned}\xi_n = E(\zeta_n \xi_{n+1}|\mathcal{F}_n) &= E(\zeta_n E(\zeta_{n+1}\xi_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n) = E(\zeta_n \zeta_{n+1} \xi_{n+2}|\mathcal{F}_n) \\ &= E(\zeta_n \cdot ... \cdot \zeta_{N-1} \xi_N|\mathcal{F}_n) = E(\eta_N|\mathcal{F}_n).\end{aligned}$$

3.3.3 If τ is a stopping time, then $\{\omega : \tau > n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ and

$$\{\omega : \tau = n\} = \{\omega : \tau > n-1\} \setminus \{\omega : \tau > n\} \in \mathcal{F}_n.$$

On the other hand if $\{\omega : \tau = n\} \in \mathcal{F}_n$ for $n = 0, 1, ...$, then

$$\{\omega : \tau \le n\} = \{\omega : \tau = 0\} \cup ... \cup \{\omega : \tau = 0\} \in \mathcal{F}_n, \quad \{\omega : \tau > n\} = \{\omega : \tau \le n\}^c \in \mathcal{F}_n.$$

Next, if τ and σ are stopping times, then, for any $n \ge 0$,

$$\{\omega : \tau \wedge \sigma > n\} = \{\omega : \tau > n\} \cap \{\omega : \sigma > n\} \in \mathcal{F}_n,$$

$$\{\omega : \tau \vee \sigma \le n\} = \{\omega : \tau \le n\} \cap \{\omega : \sigma \le n\} \in \mathcal{F}_n,$$

$$\{\omega : \tau + \sigma \le n\} = \bigcup_{i=0}^{n} \{\omega : \tau = i, \sigma = n - i\} = \bigcup_{i=0}^{n} \big(\{\omega : \tau = i\} \cap \{\omega : \sigma = n - i\}\big) \in \mathcal{F}_n.$$

Solutions for HW5 due Wed Nov 14:

3.1.4 First, as part of Fubini's theorem, we know that $g(y) := \int_{\mathbb{R}} p(x, y)\,dx$ is a Borel function of y. Therefore, $\zeta = g(\eta)$ is a random variable measurable with respect to $\sigma(\eta)$. Any Borel function of ζ is also $\sigma(\eta)$-measurable. In particular, $I_{\zeta \ne 0}\zeta^{-1}$ is $\sigma(\eta)$-measurable. In the same way we see that $h_{(\pm)}(y) := \int_{\mathbb{R}} x_{\pm} p(x, y)\,dx$ are Borel in y and $\kappa_{(\pm)} := h_{(\pm)}(\eta)$ is $\sigma(\eta)$-measurable. The product of measurable functions is measurable, and therefore $I_{\zeta \ne 0}\zeta^{-1}\kappa_{(\pm)}$ is $\sigma(\eta)$-measurable.

Next for any Borel $f(x, y) \ge 0$ we have

$$Ef(\xi, \eta) = \int_{\mathbb{R}^2} f(x, y) p(x, y)\,dxdy.$$

In particular, for any Borel $f(y) \ge 0$ we have

$$Ef(\eta) = \int_{\mathbb{R}^2} f(y) p(x, y)\,dxdy = \int_{\mathbb{R}} f(y) \big(\int_{\mathbb{R}} p(x, y)\,dx\big)\,dy = \int_{\mathbb{R}} f(y) g(y)\,dy,$$

so that g is the density of η.

Also notice that, if at a point y we have $g(y) = 0$, then $h_{(\pm)}(y) = 0$, so that for any y we have

$$I_{g(y) \ne 0}\, g^{-1}(y) h_{(\pm)}(y) g(y) = h_{(\pm)}(y).$$

Hence, for any Borel $f(y) \ge 0$,

$$Ef(\eta) I_{\zeta \ne 0}\zeta^{-1}\kappa_{(\pm)} = \int_{\mathbb{R}} I_{g(y) \ne 0} f(y) g^{-1}(y) h_{(\pm)}(y) g(y)\,dy = \int_{\mathbb{R}} f(y) h_{(\pm)}(y)\,dy$$

$$= \int_{\mathbb{R}^2} f(y) x_{\pm} p(x, y)\,dxdy = Ef(\eta)\xi_{\pm}.$$

In short,

$$Ef(\eta) I_{\zeta \ne 0}\zeta^{-1}\kappa_{(\pm)} = Ef(\eta)\xi_{\pm}. \tag{HW5.1}$$

By taking $f \equiv 1$, we get

$$EI_{\zeta \ne 0}\zeta^{-1}\kappa_{(\pm)} = E\xi_{\pm} < \infty, \quad E|\alpha| < \infty,$$

where

$$\alpha := I_{\zeta \ne 0}\zeta^{-1}\kappa_{(+)} - I_{\zeta \ne 0}\zeta^{-1}\kappa_{(-)} = I_{\zeta \ne 0}\zeta^{-1} \int_{\mathbb{R}} x p(x, \eta)\,dx$$

is $\sigma(\eta)$-measurable by the above. However, if $f = I_B$ with $B \in \mathfrak{B}(\mathbb{R})$, (HW5.1) yields

$$EI_{\eta \in B}\alpha = EI_{\eta \in B}\xi. \tag{HW5.2}$$

Now remember that α is $\sigma(\eta)$-measurable and any set in $\sigma(\eta)$ has the form $\{\omega : \eta \in B\}$ with $B \in \mathfrak{B}(\mathbb{R})$. Then we see that (HW5.2) means that $E(\xi|\eta) = \alpha$ by definition.

3.2.4(i) If $\xi_n = A_n + m_n$, where m_n is an $\mathcal{F}_n$-martingale and A_n is an increasing sequence such that A_n is $\mathcal{F}_{n-1}$-measurable for every $n \geq 2$, then (a.s.)

$$E(\xi_{n+1}|\mathcal{F}_n) = E(A_{n+1} + m_{n+1}|\mathcal{F}_n) = A_{n+1} + m_n \geq A_n + m_n = \xi_n.$$

This takes care of the "if" part.

To prove the "only if part", define

$$A_n = \sum_{i=1}^{n-1}(E(\xi_{i+1}|\mathcal{F}_i) - \xi_i), \quad n \geq 2, \quad A_1 = 0, \quad m_n = \xi_n - A_n.$$

Since $E(\xi_{i+1}|\mathcal{F}_i) - \xi_i \geq 0$ we have that A_n increases with n. Also owing to the fact that $E(\xi_{i+1}|\mathcal{F}_i) - \xi_i$ are $\mathcal{F}_{n-1}$-measurable for $i \leq n-1$, we have that A_n is $\mathcal{F}_{n-1}$-measurable. Moreover, A_n and m_n are represented as linear combinations of random variables with finite expectation. Therefore, $E|m_n| < \infty$. Finally,

$$E(m_{n+1}|\mathcal{F}_n) = E(\xi_{n+1}|\mathcal{F}_n) - A_{n+1} = E(\xi_{n+1}|\mathcal{F}_n) - \sum_{i=1}^{n}(E(\xi_{i+1}|\mathcal{F}_i) - \xi_i)$$

$$= \xi_n - \sum_{i=1}^{n-1}(E(\xi_{i+1}|\mathcal{F}_i) - \xi_i) = \xi_n - A_n = m_n.$$

6.2.2 Let ξ_t be left continuous. The other case is quite similar. Define a piecewise constant Borel function on $[0, \infty)$ by $\chi(n, 0) = 0$, $\chi(n, t) = k2^{-n}$ for $t \in (k2^{-n}, (k+1)2^{-n}]$, $k = 1, 2, ...$ Observe that

$$\xi_t^n = I_{\{0\}}(t)\xi_0 + \sum_{k=1}^{\infty} I_{(k2^{-n},(k+1)2^{-n}]}(t)\xi_{k2^{-n}},$$

where all terms are $\mathcal{F} \otimes \mathfrak{B}([0, \infty))$-measurable as products of $\mathcal{F}$- and $\mathfrak{B}([0, \infty))$-measurable functions (notice that if $f(\omega)$ is $\mathcal{F}$-measurable, then $g(\omega, t) := f(\omega)$ is $\mathcal{F} \otimes \mathfrak{B}([0, \infty))$-measurable). Therefore ξ_t^n is $\mathcal{F} \otimes \mathfrak{B}([0, \infty))$-measurable and it only remains to notice that obviously $\chi(n, t) \uparrow t$, which implies

$$\xi_t^n := \xi_{\chi(n,t)} \to \xi_t.$$

6.2.3 The indicators of measurable sets are measurable functions. Therefore, $I_{t<\tau}$ is $\mathcal{F}_t$-adapted if $\{\omega : t < \tau(\omega)\} \in \mathcal{F}_t$ for each $t \geq 0$. The latter is given by definition, so the former is true. The function $I_{t<\tau}$ is also right continuous in t. By Exercise 6.2.2 it is $\mathcal{F} \otimes \mathfrak{B}[0, \infty)$-measurable.

The function $I_{t\leq\tau}$ is left continuous in t, so to prove that it is $\mathcal{F} \otimes \mathfrak{B}[0, \infty)$-measurable it suffices to prove that it is $\mathcal{F}_t$-adapted. In turn, to do this as above it suffices to prove that

$\{\omega : t \le \tau\} \in \mathcal{F}_t$ for every $t \ge 0$. If $t = 0$, we have $\{\omega : t \le \tau\} = \Omega \in \mathcal{F}_0$. However, if $t > 0$, then

$$\{\omega : t \le \tau\} = \bigcap_{n=1}^{\infty} \{\omega : t - 1/n < \tau\} \in \mathcal{F}_t.$$

6.2.9 Let $\tau = \inf\{t \ge 0 : |\xi_t| \ge c\}$ be the first exit time of ξ_t from $(-c, c)$. By Exercise 6.2.4 we have that τ is a stopping time. Also notice that $|\xi_{T\wedge\tau}| \le c$ and $|\xi_{T\wedge\tau}| \le \sup_{t\le T} |\xi_t|$, so that

$$|\xi_{T\wedge\tau}| \le c \wedge \sup_{t\le T} |\xi_t|. \tag{HW5.3}$$

Next,

$$P(\langle\xi\rangle_T \ge N) = P(\langle\xi\rangle_T \ge N, \tau \le T) + P(\langle\xi\rangle_T \ge N, \tau > T) \le P(\tau \le T) + P(\langle\xi\rangle_T \ge N, \tau > T).$$

Here

$$P(\tau \le T) = P(\sup_{t\le T} |\xi_t| \ge c),$$

and owing to Chebyshev, Theorem 6.2.7, and and (HW5.3),

$$P(\langle\xi\rangle_T \ge N, \tau > T) \le P(\langle\xi\rangle_{T\wedge\tau} \ge N) \le N^{-2} E\langle\xi\rangle^2_{T\wedge\tau} = N^{-2} E\xi^2_{T\wedge\tau}$$
$$\le N^{-2} E\big(c \wedge \sup_{t\le T} |\xi_t|\big)^2.$$

Combining these estimates, we get what we need.

6.2.10 We only prove the first inequality. The second one is proved similarly. From Exercise 6.2.9 for $c > 0$ we have

$$P(\langle\xi\rangle_T^{1/2} \ge c) \le \frac{1}{c^2} E\big(c^2 \wedge \eta^2\big) + P(\eta \ge c), \quad \eta := \sup_{t\le T} |\xi_t|. \tag{HW5.4}$$

Now we use again the fact that for $\zeta \ge 0$

$$E\zeta = \int_0^\infty P(\zeta \ge c)\, dc.$$

Then by integrating in (HW5.4) we find that

$$E\langle\xi\rangle_T^{1/2} \le E\eta + E\int_0^\infty \frac{1}{c^2}(c^2 \wedge \eta^2)\, dc = E\eta + E\int_0^\eta dc + E\int_\eta^\infty \frac{\eta^2}{c^2}\, dc = 3E\eta.$$

Solutions for HW6 due Wed Nov 28:

6.3.6 In exactly the same way as Theorem 5 (ii) one derives from Exercise 6.3.9 that

$$P\Big(\int_0^T |f_s - f_s^n|^2\, ds \ge N\Big) \le \frac{c^2}{N} + P\Big(\sup_{t\le T} \Big|\int_0^t f_s\, dw_s - \int_0^t f_s^n\, dw_s\Big| \ge c\Big). \tag{HW6.1}$$

If

$$\sup_{t\le T} \Big|\int_0^t f_s^n\, dw_s - \int_0^t f_s\, dw_s\Big| \xrightarrow{P} 0,$$

then (HW6.1) implies that for any $c > 0$

$$\overline{\lim_{n\to\infty}} P\Big(\int_0^T |f_s - f_s^n|^2\, ds \ge N\Big) \le \frac{c^2}{N}.$$

Since c is arbitrary, the left-hand side is zero and we are done.

6.3.12 For $t < 1$ define

$$y(t) = \int_0^t \frac{1}{1-s}\, dw_s, \quad \phi(t) = \int_0^t \frac{1}{(1-s)^2}\, ds = \frac{t}{1-t}, \quad \psi(r) = \frac{r}{r+1}.$$

Notice that $\phi(\psi(r)) \equiv r$. From the proof of Theorem 2.4.2 we know that $y(t), t < 1$ admits a continuous modification, namely,

$$w_t(1-t)^{-1} - \int_0^t w_s(1-s)^{-2}\, ds,$$

and $y(\psi(r))$, $r \ge 0$, is a Wiener process. The Wiener process hits any point with probability one. Therefore, with probability one $\sigma := \inf\{r \ge 0 : y(\psi(r)) = -1\} < \infty$. Hence, $\tau = \psi(\sigma) < 1$ (a.s.). In particular,

$$\int_0^\infty I_{s<\tau}(1-s)^{-2}\, ds < \infty \quad \text{(a.s.)}. \tag{HW6.2}$$

Furthermore, $y(t)$ is continuous and $\mathcal{F}_t$-adapted. Therefore, τ is a stopping time. It follows that $I_{s<\tau}(1-s)^{-1} \in \mathcal{S}$ and the process

$$\eta_t := \int_0^t I_{s<\tau}(1-s)^{-1}\, dw_s$$

is well defined.

Next, for $n \ge 2$ introduce $t(n) = 1 - 1/n$. Observe that $t(n) \uparrow 1$. Then owing to (HW6.2) and the fact that $\tau < 1$, the dominated convergence theorem implies that

$$\int_0^\infty |I_{s<\tau} - I_{s<\tau\wedge t(n)}|^2 (1-s)^{-2}\, ds \to 0 \quad \text{(a.s.)}.$$

It follows that the processes

$$\eta_t(n) := \int_0^t I_{s<\tau\wedge t(n)}(1-s)^{-1}\, dw_s$$

converge to η_t uniformly in t on any finite time interval in probability.

Now, $I_{s<t(n)}(1-s)^{-1} \in H$, so that by Theorem 6.3.7 (iv) (a.s.) for $t \ge 1$ (remember $1 > \tau$)

$$\eta_t(n) = \int_0^t I_{s<\tau} I_{s<t(n)}(1-s)^{-1}\, dw_s$$

$$= \int_0^{t\wedge\tau} I_{s<t(n)}(1-s)^{-1}\, dw_s = \xi_{t\wedge\tau}(n) = \xi_\tau(n),$$

where

$$\xi_r(n) := \int_0^r I_{s<t(n)}(1-s)^{-1}\, dw_s.$$

Next, for any particular $r \geq 0$ (a.s.) (notice $t(n) < 1$)

$$\xi_r(n) = \int_0^\infty I_{s<t(n)\wedge r}(1-s)^{-1}\,dw_s = \int_0^\infty I_{s<1}I_{s<t(n)\wedge r}(1-s)^{-1}\,dw_s$$
$$= \int_0^1 I_{s<t(n)\wedge r}(1-s)^{-1}\,dw_s = y(t(n)\wedge r).$$

In short, $\xi_r(n) = y(t(n)\wedge r)$ (a.s.) for any r. Since both parts are continuous in r, they coincide for all r at once on a set of probability 1. Therefore, $\xi_\tau(n) = y(t(n)\wedge\tau)$ (a.s.). Finally, for $t \geq 1$ (a.s.)

$$\eta_t = P\text{-}\lim_{n\to\infty}\eta_t(n) = \lim_{n\to\infty}\xi_\tau(n) = \lim_{n\to\infty} y(t(n)\wedge\tau) = y(\tau) = -1,$$

which is exactly what we need.

Comment. Some of you believed that

$$E\int_0^\infty I_{s<\tau}(1-s)^{-2}\,ds < \infty.$$

However, $\tau = \psi(\sigma)$, $\phi(\tau) = \sigma$, and

$$E\int_0^\infty I_{s<\tau}(1-s)^{-2}\,ds = E\phi(\tau) = E\sigma = \infty,$$

where the last equality can be obtained from the distribution of σ (Wald's distribution) or from Remark 6.3.10.

6.3.13 Let

$$\xi_t = \int_0^t f_s\,dw_s, \quad \tau(n) = \inf\{t \geq 0 : \int_0^t f_s^2\,ds \geq n\}.$$

Then $E\int_0^\infty I_{t\leq\tau(n)}f_t^2\,dt \leq n$ and by Theorem 6.3.8 (ii)

$$\xi_{t\wedge\tau(n)} = \int_0^t I_{s\leq\tau(n)}f_s\,dw_s$$

is a martingale. By Davis's inequality

$$E\sup_{t\leq T}|\xi_{t\wedge\tau(n)}| \leq 3E\Big(\int_0^T I_{t\leq\tau(n)}f_t^2\,dt\Big)^{1/2} \leq 3E\Big(\int_0^T f_t^2\,dt\Big)^{1/2}.$$

By letting $n\to\infty$ and using Fatou's theorem we get

$$E\sup_{t\leq T}|\xi_t| \leq E\Big(\int_0^T f_t^2\,dt\Big)^{1/2}.$$

This allows to use the dominated convergence theorem for conditional expectations and pass to the limit as $n\to\infty$ in

$$E(\xi_{t\wedge\tau(n)}|\mathcal{F}_s) = \xi_{s\wedge\tau(n)} \quad \text{(a.s.)} \quad s \leq t \leq T.$$

Then the result follows.

6.7.4 (i) For $r \geq 0$ define $\tau(r) = \inf\{t \geq 0 : |w_t - x_0| = r\}$. For $0 < \varepsilon \leq |x_0|$ obviously τ_ε is a decreasing function of ε. Therefore, the limit

$$\tau := \lim_{\varepsilon\downarrow 0}\tau_\varepsilon$$

exists. Obviously τ is the first time w_t hits x_0. Since each trajectory of w_t is continuous, it is bounded on $[0,\tau]$ if $\tau(\omega)<\infty$. Therefore, for almost any trajectory there exists $n>|x_0|$ such that $\tau<\tau_n$. In other words,

$$\{\omega:\tau<\infty\}\subset\bigcup_{n>|x_0|}\{\omega:\tau<\tau_n\}.$$

The sets on the right are nested, hence

$$P(\tau<\infty)\leq\lim_{n\to\infty}P(\tau<\tau_n). \tag{HW6.3}$$

Since $\tau>\tau_\varepsilon$ for $\varepsilon<|x_0|$, it follows from Example 6.7.2 that, for $n>|x_0|$, we have

$$P(\tau<\tau_n)\leq\varlimsup_{\varepsilon\downarrow 0}P(\tau_\varepsilon<\tau_n)=0.$$

Now (HW6.3) implies that $\tau=\infty$ (a.s.), so that $\tau_\varepsilon\to\infty$ (a.s.) indeed.

(ii) By taking any smooth function f such that $f(x)=\ln|x-x_0|$ for $|x-x_0|\geq\varepsilon$ in the same way as in Example 3 we get

$$\ln|w_{t\wedge\tau_\varepsilon}-x_0|=\ln|x_0|+\int_0^t I_{s\leq\tau_\varepsilon}|w_s-x_0|^{-2}(w_s-x_0)\,dw_s.$$

Since with probability 1 there is no t such that $w_t=x_0$ ($\tau=\infty$), we get that for almost any ω the function $|w_s-x_0|^{-2}$ is well defined as a function of t is continuous and hence bounded on each finite time interval. Therefore,

$$\int_0^T|w_s-x_0|^{-4}|w_s-x_0|^2\,ds<\infty\quad\text{(a.s.)}$$

for any $T<\infty$. Therefore, the integral $\int_0^t|w_s-x_0|^{-2}(w_s-x_0)\,dw_s$ is well defined and by the properties of stochastic integrals on $\mathcal{S}$ we conclude (a.s.)

$$\ln|w_{t\wedge\tau_\varepsilon}-x_0|=\ln|x_0|+\int_0^{t\wedge\tau_\varepsilon}|w_s-x_0|^{-2}(w_s-x_0)\,dw_s,$$

which for $\varepsilon\downarrow 0$ implies that (a.s.)

$$\ln|w_t-x_0|=\ln|x_0|+\int_0^t|w_s-x_0|^{-2}(w_s-x_0)\,dw_s. \tag{HW6.4}$$

(iii) If the stochastic integral in (HW6.4) were a martingale, its expectation would be zero, that is, we would have

$$\ln|x_0|=E\ln|w_t-x_0|. \tag{HW6.5}$$

To get a contradiction, use the hint and for $x\in\mathbb{R}^2$ denote

$$\phi_n(x):=\int_n^{n|x|^{-2}}s^{-1}e^{-1/(2s)}\,ds=\int_0^{n|x|^{-2}}-\int_0^n=\int_0^n s^{-1}e^{-|x|^2/(2s)}\,ds-\int_0^n s^{-1}e^{-1/(2s)}\,ds,$$

and notice that

$$|\phi_n(x)|\leq\Big|\int_n^{n|x|^{-2}}s^{-1}\,ds\Big|=2|\ln|x||,$$

$$\Big|\int_n^{n|x|^{-2}}s^{-1}e^{-1/(2s)}\,ds-\int_n^{n|x|^{-2}}s^{-1}\,ds\Big|\leq\int_n^{n|x|^{-2}}s^{-2}\,ds=0,$$

$$\lim_{n\to\infty}\phi_n(x)=\lim_{n\to\infty}\int_n^{n|x|^{-2}} s^{-1}\,ds=-2\ln|x|.$$

Hence by the dominated convergence theorem

$$2E\ln|w_t-x_0|=\lim_{n\to\infty}\big[\int_0^n s^{-1}e^{-1/(2s)}\,ds-\int_0^n s^{-1}Ee^{-|w_t-x_0|^2/(2s)}\,ds\big].$$

By using the fact that the density of a sum of independent random variables is the convolution of their densities and that the sum of Gaussian variables is Gaussian we get without computation that

$$s^{-1}Ee^{-|w_t-x_0|^2/(2s)}=(s+t)^{-1}e^{-|x_0|^2/(s+t)}.$$

Hence

$$2E\ln|w_t-x_0|=\lim_{n\to\infty}\big[\int_0^n s^{-1}e^{-1/(2s)}\,ds-\int_0^n (s+t)^{-1}e^{-|x_0|^2/(s+t)}\,ds\big].$$

By noticing that

$$\int_n^{n+t} s^{-1}e^{-1/(2s)}\,ds\le\int_n^{n+t} s^{-1}\,ds=\ln(1+t/n)\to 0$$

as $n\to\infty$, we finally conclude

$$2E\ln|w_t-x_0|=\lim_{n\to\infty}\big[\int_0^n s^{-1}e^{-1/(2s)}\,ds-\int_{t|x_0|^{-2}}^{(n+t)|x_0|^{-2}} s^{-1}e^{-1/(2s)}\,ds\big]$$

$$=\lim_{n\to\infty}\big[\int_0^{n+t} s^{-1}e^{-1/(2s)}\,ds-\int_0^{(n+t)|x_0|^{-2}} s^{-1}e^{-1/(2s)}\,ds\big]+\int_0^{t|x_0|^{-2}} s^{-1}e^{-1/(2s)}\,ds$$

$$=2\ln|x_0|+\int_0^{t|x_0|^{-2}} s^{-1}e^{-1/(2s)}\,ds>2\ln|x_0|,$$

which contradicts (HW6.5) indeed.

6.8.4 (ii) Define

$$\tau(n)=\inf\{t\ge 0:\int_0^t \rho_s^2b_s^2\,ds\ge n\}.$$

In the proof of Lemma 6.8.1 it is noticed that $\rho_{t\wedge\tau(n)}$ is a martingale. It is also a nonnegative submartingale. By Doob's inequality

$$E\sup_{t\le T}\rho^p_{t\wedge\tau(n)}\le (p/(p-1))^pE\rho^p_{T\wedge\tau(n)}\le (p/(p-1))^pN.$$

By letting $n\to\infty$, using Fatou and Hölder, we conclude that $E\sup_{t\le T}\rho_t<\infty$. We are now done due to the first part of this exercise.

6.8.5 Take a stopping time $\tau\le T$ and observe that, for any $\alpha\in(0,1)$ and p,

$$E\rho^p_\tau=E(\xi^\alpha\eta^{1-\alpha})\le(E\xi)^\alpha(E\eta)^{1-\alpha},\qquad\text{(HW6.6)}$$

where

$$\xi=\exp\big(p\alpha^{-1}\int_0^\tau b_s\,dw_s-p^22^{-1}\alpha^{-2}\int_0^\tau|b_s|^2\,ds\big),\quad \eta=\exp\big(k(p,\alpha)\int_0^\tau|b_s|^2\,ds\big),$$

and

$$k(p,\alpha)=\frac{p}{2}\big(\frac{p}{\alpha}-1\big)\frac{1}{1-\alpha}.$$

Notice that $\xi = \rho_\tau(p\alpha^{-1}b)$ and ρ_t is a supermartingale, so that by Corollary 3.3.12 we have $E\rho_\tau \le E\rho_0 = 1$. Of course, one should say that Corollary 3.3.12 was only proved for discrete time and that in our case one has first to approximate τ with discrete stopping times and then pass to the limit on the basis of Fatou's theorem.

Thus (HW6.6) implies that

$$E\rho_\tau^p \le \big(E\exp\big(k(p,\alpha)\int_0^\tau |b_s|^2\,ds\big)\big)^{1-\alpha} \le \big(E\exp\big(k(p,\alpha)\int_0^T |b_s|^2\,ds\big)\big)^{1-\alpha}$$

for any $\alpha \in (0,1)$ and $p > 1$. Next, notice that $k(p,\alpha)$ tends to $1/2$ at $\alpha \to 1$ and $p = 1 + o(1-\alpha)$. Therefore, there exist $\alpha \in (0,1)$ and $p > 1$ such that $k(p,\alpha) \le c$. After that it only remains to refer to Exercise 6.8.4 (ii).

Solutions for HW 7 due Wed Dec 12:

6.8.6 By Holder's inequality

$$E\rho_T((1-\varepsilon)b) \le \big[E\rho_T(b)\big]^{1-\varepsilon}\big[E\exp\frac{1-\varepsilon}{2}\int_0^T |b_t|^2\,dt\big]^{\varepsilon}. \qquad \text{(HW7.1)}$$

If condition (4) is satisfied, then

$$E\exp\frac{1-\varepsilon}{2}\int_0^T |b_t|^2\,dt < \infty$$

for small $\varepsilon > 0$ implying that

$$E\exp\frac{1+\varepsilon}{2}\int_0^T (1-\varepsilon)^2|b_t|^2\,dt \le E\exp\frac{1-\varepsilon^2}{2}\int_0^T |b_t|^2\,dt < \infty.$$

Now Exercise 5 and (HW7.1) lead to

$$1 \le \big[E\rho_T(b)\big]^{1-\varepsilon}\big[E\exp\frac{1-\varepsilon}{2}\int_0^T |b_t|^2\,dt\big]^{\varepsilon},$$

$$0 \le (1-\varepsilon)E\ln\rho_T(b) + \varepsilon\ln E\exp\frac{1-\varepsilon}{2}\int_0^T |b_t|^2\,dt$$

By letting $\varepsilon \downarrow 0$ and using assumption (4), we get $E\rho_T(b) \ge 1$. However, always $E\rho_T(b) \le 1$. Hence $E\rho_T(b) = 1$.

Finally, if $E\exp(1/2)\int_0^T |b_t|^2\,dt < \infty$, then (4) is satisfied since

$$\ln E\exp\frac{1-\varepsilon}{2}\int_0^T |b_t|^2\,dt \le \ln E\exp(1/2)\int_0^T |b_t|^2\,dt.$$

6.10.1 Take a continuous function $f(x)$ on $[r,1]$ and solve the equation

$$a(x)u''(x) + b(x)u'(x) = -f(x)$$

on $[r,1]$ with zero condition at the endpoints of this interval. To this end denote $v = u'$ and observe that

$$v' + (b/a)v = -f/a, \quad \phi(v\phi^{-1})' = -f/a, \quad (v\phi^{-1})' = -\phi^{-1}f/a,$$

$$v(x) = \phi(x)\int_x^1 \phi^{-1}(s)f(s)/a(s)\,ds + c_1\phi(x),$$

$$u(x) = \int_r^x \phi(t)\big(\int_t^1 \phi^{-1}(s)f(s)/a(s)\,ds\big)\,dt + c_1\int_r^x \phi(t)\,dt + c_2,$$

where c_i are some constants which we find from the conditions $u(r) = u(1) = 0$. The condition $u(r) = 0$ yields $c_2 = 0$. Next we use Fubini's theorem to get

$$u(x) = \int_r^1 f(s)a^{-1}(s)\phi^{-1}(s)\big(\int_r^{s\wedge x} \phi(t)\,dt\big)\,ds + c_1\int_r^x \phi(t)\,dt.$$

Now the condition $u(1) = 0$ shows that

$$c_1 = -\psi^{-1}\int_r^1 f(s)a^{-1}(s)\phi^{-1}(s)\big(\int_r^s \phi(t)\,dt\big)\,ds$$

and hence

$$u(x) = \int_r^1 f(s)a^{-1}(s)\phi^{-1}(s)\big(\int_r^{s\wedge x} \phi(t)\,dt - \psi^{-1}\int_r^s \phi(t)\,dt\int_r^x \phi(t)\,dt\big)\,ds.$$

Finally we notice that, if $s \le x$, then

$$q(s,x) := \psi\int_r^{s\wedge x} \phi(t)\,dt - \int_r^s \phi(t)\,dt\int_r^x \phi(t)\,dt = \int_r^s \phi(t)\,dt\big(\psi - \int_r^x \phi(t)\,dt\big)$$

$$= \int_r^s \phi(t)\,dt\int_x^1 \phi(t)\,dt = \int_r^{s\wedge x} \phi(t)\,dt\int_{s\vee x}^1 \phi(t)\,dt.$$

If $s \ge x$, then

$$q(s,x) = \psi\int_r^x \phi(t)\,dt - \int_r^s \phi(t)\,dt\int_r^x \phi(t)\,dt = \int_r^x \phi(t)\,dt\big(\psi - \int_r^s \phi(t)\,dt\big)$$

$$= \int_r^{s\wedge x} \phi(t)\,dt\int_{s\vee x}^1 \phi(t)\,dt.$$

Thus our solution is given by the formula in question and hence the function it defines possesses indeed the asserted properties.

6.10.3 (i) Take the function u from the hint and notice that by Itô's formula

$$u(\xi_t) = u(0) + \int_0^t [u'(\xi_s)b(\xi_s) + (1/2)\sigma^2(\xi_s)u''(\xi_s)]\,ds + \int_0^t \sigma(\xi_s)u'(\xi_s)\,dw_s$$

$$= \int_0^t u'(\xi_s)b(\xi_s)\,ds,$$

where the second equality is true (a.s.) because $u(0) = 0$ and $\sigma u' = \sigma u'' \equiv 0$. Since $u' \le 0$ and $b \ge 0$, we conclude $u(\xi_t) \le 0$ (a.s.). However, $u \ge 0$. Hence $u(\xi_t) = 0$ (a.s.), which implies that $\xi_t \ge c$ (a.s.).

(ii) We have

$$\xi_{t\wedge\tau(r)} = \int_0^{t\wedge\tau(r)} b(\xi_s)\,ds + \int_0^{t\wedge\tau(r)} \sigma)\xi_s)\,dw_s.$$

Since σ is bounded on $[r,1]$, by taking expectations we conclude

$$E\xi_{t\wedge\tau(r)} = E\int_0^{t\wedge\tau(r)} b(\xi_s)\,ds. \tag{HW7.2}$$

Now let $t\to\infty$. Then the right-hand side converges to $E\int_0^{\tau(r)} b(\xi_s)\,ds$ by the monotone convergence theorem. Since $\xi_{t\wedge\tau(r)}\le 1$ and $\delta>0$, we see that

$$\delta E\tau(r)\le E\int_0^{\tau(r)} b(\xi_s)\,ds = \lim_{t\to\infty} E\int_0^{t\wedge\tau(r)} b(\xi_s)\,ds\le 1,\quad \tau(r)<\infty\quad \text{(a.s.)}.$$

Therefore $\xi_{t\wedge\tau(r)}\to\xi_{\tau(r)}$ as $t\to\infty$ and by (i) we have $\xi_{\tau(r)}=1$ (a.s.). Thus by the dominated convergence theorem the left-hand side of (HW7.2) goes to 1 and we are done.

6.10.4 We again use (HW7.2) and as above obtain that $\tau(r)<\infty$ (a.s.) and let $t\to\infty$. Then we get

$$E\tau(r)=E\xi_{\tau(r)}=1\cdot P(\xi_{\tau(r)}=1)+r\cdot P(\xi_{\tau(r)}=r).$$

Since $r<0$ we conclude $E\tau(r)\le 1$.

4.1.15 As usual we may assume that $R(0)=1$. Owing to Cauchy's criterion we onle need to check that

$$E\Big|\frac{1}{2T}\int_{-T}^{T}\xi_t\,dt-\frac{1}{2S}\int_{-S}^{S}\xi_t\,dt\Big|^2\to 0 \tag{HW7.3}$$

as $T,S\to\infty$. By definition the left-hand side of (HW7.3) equals the limit as $n\to\infty$ of

$$E\Big|\frac{1}{2T}\sum_{k=-T2^n}^{T2^n}\xi_{k2^{-n}}2^{-n}-\frac{1}{2S}\sum_{k=-S2^n}^{S2^n}\xi_{k2^{-n}}2^{-n}\Big|^2. \tag{HW7.4}$$

Simple arithmetical manipulations show that (HW7.4) is expressed through the correlation function alone and thus will not change if we take a different process with the same correlation function. As usual we prefer to deal with $\xi_t=e^{i(\xi t+\phi)}$, where the random variable ξ has the characteristic function equal to R. We substitute this ξ_t into (HW7.4) and observe that, due to the dominated convergence theorem and the fact that $|\xi_t|\le 1$, as $n\to\infty$, the new expressions (HW7.4) tend to the left-hand side of (HW7.3), where the integrals are just Riemann integrals. After that we notice that ϕ disappears due to $|\cdot|$ and

$$\frac{1}{2T}\int_{-T}^{T}e^{i\xi t}\,dt = I_{\xi\ne 0}\frac{\sin T\xi}{T\xi}+I_{\xi=0}.$$

Then we see that we only have to prove that

$$E\Big|I_{\xi\ne0}\frac{\sin T\xi}{T\xi}-I_{\xi\ne0}\frac{\sin S\xi}{S\xi}\Big|\to 0 \tag{HW7.5}$$

as $T,S\to\infty$. However (HW7.5) trivially follows from the dominate convergence theorem since $|\sin x/x|\le 1$ and $|\sin x/x|\to 0$ as $|x|\to\infty$. This take care of the first part of the problem.

To investigate when

$$E\Big|\frac{1}{2T}\int_{-T}^{T}\xi_t\,dt\Big|^2\to 0 \tag{HW7.6}$$

we rewrite the expression on the left in terms of $\xi_t = e^{i(\xi t+\phi)}$ as

$$E\big|I_{\xi\neq 0}\frac{\sin T\xi}{T\xi} + I_{\xi=0}\big|^2 = P(\xi = 0) + E\big|I_{\xi\neq 0}\frac{\sin T\xi}{T\xi}\big|^2.$$

Again the last term tends to zero and we see that (HW7.6) holds if and only if $P(\xi = 0) = 0$, which is exactly what we had to show.

To do the last part of the problem we use the hint showing that $F(\{0\}) = 0$ if and only if

$$\frac{1}{T}\int_0^T R(t)\,dt \to 0.$$

We are given that $R(t) \to 0$ as $t \to \infty$. Therefore, for any $\varepsilon > 0$ there is a $T_0 < \infty$ such that $|R(t)| \leq \varepsilon$ for all $t \geq T_0$. Therefore,

$$\overline{\lim_{T\to\infty}}\,\big|\frac{1}{T}\int_0^T R(t)\,dt\big| \leq \overline{\lim_{T\to\infty}}\,\big|\frac{1}{T}\int_0^{T_0} R(t)\,dt\big| + \overline{\lim_{T\to\infty}}\,\big|\frac{1}{T}\int_{T_0}^T R(t)\,dt\big| \leq \varepsilon\,\overline{\lim_{T\to\infty}}\,\frac{T-T_0}{T} \leq \varepsilon.$$

We get the result we need since ε is arbitrary.

4.2.8 Corollary 4.2.7 implies that

$$\int_{\mathbb{R}} \bar{f}(x)\,\zeta(dx) = \overline{\int_{\mathbb{R}} f(x)\,\zeta(dx)} = \int_{\mathbb{R}} \bar{f}(-x)\,\zeta(dx) \quad \text{(a.s.)} \quad \forall f \in L_2(\mathfrak{B}(\mathbb{R}), F).$$

For $f(x) = e^{itx}$ this yields $\xi_t = \xi_{-t}$. Then $R(2t) = E\xi_t\xi_{-t} = E\xi_t^2 = R(0)$ and

$$E(\xi_{2t} - \xi_0)^2 = 2R(0) - 2E\xi_{2t}\xi_0 = 2R(0) - 2R(2t) = 0,$$

so that $\xi_{2t} = \xi_0$ (a.s.) for any t as required.

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3.3.13 Use the hint. Namely, fix n and $A \in \mathcal{F}_n$ and define $\tau(\omega) = n + 1$ if $\omega \in A$ and $\tau(\omega) = n + 1$ if $\omega \notin A$. Then $\{\omega : \tau = i\}$ equals $\emptyset \in \mathcal{F}_i$ if $i \neq n, n+1$. However, if $i = n$, then $\{\omega : \tau = i\} = A \in \mathcal{F}_n = \mathcal{F}_i$, and if $i = n + 1$, then $\{\omega : \tau = i\} = A^c \in \mathcal{F}_n \subset \mathcal{F}_i$. Therefore $\{\omega : \tau = i\} \in \mathcal{F}_i$ for all i and τ is a stopping time. Also $\tau \leq \sigma := n + 1$ and the inequality $E\xi_\tau \leq E\xi_\sigma$ becomes

$$EI_A\xi_n + E_{A^c}\xi_{n+1} \leq E\xi_{n+1}, \quad EI_A\xi_n \leq E\xi_{n+1} - E_{A^c}\xi_{n+1} = EI_A\xi_{n+1}.$$

Hence $EI_A\xi_n \leq E[I_A E(\xi_{n+1}|\mathcal{F}_n)]$ and the arbitrariness of $A \in \mathcal{F}_n$ implies that $\xi_n \leq E(\xi_{n+1}|\mathcal{F}_n)$, which is exactly what we need.

4.3.3 "$\Longrightarrow$" Let $R(t)$ be the correlation function of ξ_t. Assume without losing generality that $R(0) = 1$. Then, for any s the random variables $\xi_{t+s} - R(s)\xi_t$ and ξ_t are uncorrelated and hence independent. In particular, (a.s.)

$$E(\xi_{t+s}|\xi_t) = E(\xi_{t+s} - R(s)\xi_t|\xi_t) + R(s)\xi_t = R(s)\xi_t.$$

By adding the Markov property, we conclude that for $s, t \geq 0$

$$R(t + s) = E\xi_{t+s}\xi_0 = E[\xi_0 E(\xi_{t+s}|\xi_0, \xi_t)] = E\xi_0 E(\xi_{t+s}|\xi_t) = R(s)E\xi_0\xi_t = R(s)R(t).$$

In short, $R(t + s) = R(s)R(t)$. One knows that all continuous solutions of this equation are exponential functions. Since $R(0) = 1$, we have $R(t) = e^{-\alpha t}$, $t \geq 0$, and since R is bounded, $\alpha \geq 0$. Finally, ξ_t is real, $R(-t) = R(t)$, and $R(t) = e^{-\alpha|t|}$ for all t.

"$\Longleftarrow$" We only need prove the Markov propery of ξ_t. Again we assume that $R(0) = 1$. Then for any $s \geq 0$ and $r \leq t$ we have

$$E[\xi_{t+s} - R(s)\xi_t]\xi_r = R(t+s-r) - R(s)R(t-r) = e^{-\alpha(t+s-r)} - e^{-\alpha s}e^{-\alpha(t-r)} = 0.$$

Since the process ξ_t is Gaussian, it follows that the random variable $\xi_{t+s} - R(s)\xi_t$ and the process ξ_r, $r \leq t$ are independent. Now use Theorem 3.1.13 to conclude that for any bounded Borel f and $t_1 \leq ... \leq t_n = t$ we have

$$E\{f(\xi_{t+s})|\xi_{t_1}, ..., \xi_{t_n}\} = E\{f(\xi_{t+s} - R(s)\xi_t + R(s)\xi_t)|\xi_{t_1}, ..., \xi_{t_n}\} = \Phi(\xi_t)$$

(a.s.), where Φ is a Borel function. Finally, by taking conditional expectations given ξ_t of the extreme terms and remembering Theorem 3.6.1 (v), we conclude (a.s.)

$$E\{f(\xi_{t+s})|\xi_t\} = \Phi(\xi_t).$$

Hence (a.s.)

$$E\{f(\xi_{t+s})|\xi_{t_1}, ..., \xi_{t_n}\} = \Phi(\xi_t) = E\{f(\xi_{t+s})|\xi_t\}$$

and we are almost done. It only remains to invoke standard arguments allowing one to pass to general f is question from bounded ones.

4.4.3 Use the hint and let $\tilde{\varphi}$ be the function obtained from φ by replacing all the coefficients of P_+ and Q_+ with their conjugate. Next, define two analytic functions $\psi_1(z) = \tilde{\varphi}(z)$ and $\psi_2(z) = \varphi(-z)$. The equality $\bar{\varphi}(x) = \varphi(-x)$ for real x means that $\psi_1 = \psi_2$ on the real line. Therefore, $\psi_1(z) = \psi_2(z)$ for all z for which the functions are defined. In particular, they coincide on the imaginary axis apart from finitely number of possible poles. Finally, for real x, obviously, $\psi_1(ix) = \overline{\varphi(-ix)}$. Thus

$$\overline{\varphi(-ix)} = \varphi(-ix),$$

which means that $\varphi(-ix)$ is real for $x \in \mathbb{R}$ and hence $\varphi(ix)$ is real too.

6.8.10 First, Corollary 6.8.3 with $b_s = \pm K I_{s \leq \tau}$ and t such that $\tau \leq t$ implies that

$$Ee^{\pm K w_\tau} < \infty, \quad Ee^{K|w_\tau|} < \infty \quad \forall K < \infty. \tag{Fn.1}$$

Furthermore, formal multiple differentiation of $Ee^{\lambda w_\tau - \lambda^2\tau/2}$ with respect to λ produces the expectation of expressions like

$$P(w_\tau, \tau)e^{\lambda w_\tau - \lambda^2\tau/2},$$

where P is a polynomial. Since τ bounded, such expressions by magnitude are less than

$$N(1 + |w_\tau|^m)e^{\lambda|w_\tau|} \leq Ne^{(1+\lambda)|w_\tau|}, \tag{Fn.2}$$

where N's and m are certain finite nonrandom constants. Owing to (Fn.1) the expectation of (Fn.2) is a locally bounded function of λ. This and a well-known rule of differentiating integrals with respect to parameters proves that indeed we can differentiate the equality

$$Ee^{\lambda w_\tau - \lambda^2\tau/2} = 1$$

bringing all derivatives inside the expectation sign. Thus,

$$E\frac{d^{2k}}{(d\lambda)^{2k}}e^{\lambda w_\tau - \lambda^2\tau/2}\Big|_{\lambda=0} = 0. \tag{Fn.3}$$

To transform the left-hand side of (Fn.3) use the hint to the exercise. Notice that

$$e^{-\nu^2}=\sum_{n=0}^{\infty}\frac{1}{n!}\,(-1)^n\nu^{2n},$$

which shows that all odd-order derivatives of $e^{-\nu^2}$ vanish at zero and

$$\frac{d^{2n}}{(d\nu)^{2n}}e^{-\nu^2}\big|_{\nu=0}=(-1)^n.$$

Hence

$$f_\mu^{(2k)}(r,0)=\sum_{n=0}^{2k}\binom{2k}{n}\big[\frac{d^n}{(d\mu)^n}e^{\mu r}\big]\frac{d^{2k-n}}{(d\mu)^{2k-n}}e^{-\mu^2/2}\big|_{\mu=0}=\sum_{n=0}^{k}b_nr^{2n},$$

where

$$b_0=\frac{d^{2k}}{(d\mu)^{2k}}e^{-\mu^2/2}\big|_{\mu=0}\neq 0,\quad b_k=1.$$

Next,

$$\frac{d^{2k}}{(d\lambda)^{2k}}e^{\lambda w_\tau-\lambda^2\tau/2}\big|_{\lambda=0}=f_\mu^{(2k)}(r,0)(\mu'_\lambda)^{2k}=\sum_{n=0}^{k}b_nr^{2n}\tau^k=\sum_{n=0}^{k}b_nw_\tau^{2n}\tau^{k-n}.$$

We substitute this in (Fn.3) and get

$$E(b_kw_\tau^{2k}+b_{k-1}w_\tau^{2k-2}\tau+b_{k-2}w_\tau^{2k-4}\tau^2+...+b_0\tau^k)=0.$$

Now use that for any $p,q\geq 0$ and $\varepsilon>0$ there exists a constant N such that for all $x,y\geq 0$ we have

$$x^py^q\leq\varepsilon x^{p+q}+Ny^{p+q}.$$

By the way, below by ε and N we denote small and large constants which may change from one appearence to another. This elementary inequality implies in particular that

$$Ew_\tau^{2k-2n}\tau^n=Ew_\tau^{2k-2n}(\sqrt{\tau})^{2n}\leq\varepsilon Ew_\tau^{2k}+NE\tau^k,$$

$$Ew_\tau^{2k-2n}\tau^n\leq\varepsilon E\tau^k+NEw_\tau^{2k}.\tag{Fn.4}$$

Then to estimate Ew_τ^{2k} we write

$$Ew_\tau^{2k}=-b_k^{-1}[b_{k-1}Ew_\tau^{2k-2}\tau+b_{k-2}Ew_\tau^{2k-4}\tau^2+...+b_0E\tau^k]$$
$$=c_{k-1}Ew_\tau^{2k-2}\tau+c_{k-2}Ew_\tau^{2k-4}\tau^2+...+c_0E\tau^k$$
$$\leq|c_{k-1}|Ew_\tau^{2k-2}\tau+c_{k-2}Ew_\tau^{2k-4}\tau^2+...+|c_0|E\tau^k\leq\varepsilon Ew_\tau^{2k}+NE\tau^k.$$

In short, for any $\varepsilon>0$ there is a constant N such that $Ew_\tau^{2k}\leq\varepsilon Ew_\tau^{2k}+NE\tau^k$. Upon taking $\varepsilon=1/2$, we get $Ew_\tau^{2k}\leq NE\tau^k$.

To estimate $E\tau^k$ through Ew_τ^{2k} we use the second inequality in (Fn.4) while starting as follows

$$E\tau^k=-b_0^{-1}[b_kEw_\tau^{2k}+b_{k-1}Ew_\tau^{2k-2}\tau+b_{k-2}Ew_\tau^{2k-4}\tau^2+...+b_1Ew_\tau^2\tau^{k-1}].$$