

Homework 1 due on Monday September 8, 2008

Chapter 1, Problems 4, 5, 7.

Also solve the following problems

(A) Let \mathcal{E} be the collection of intervals in $\mathbb{R} = (-\infty, \infty)$ of type (r, ∞) where r is an arbitrary rational number. Prove that $\sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R})$.

(B) Let \mathcal{E} be the collection of open balls in \mathbb{R}^d . Show that $\sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R}^d)$. (You may need to recall that the set of points in \mathbb{R}^d with rational coordinates is countable.)

Ordered results for HW1 (out of 35 pts):

34, 34, 31, 31, 31, 30.5, 30, 30, 30, 30, 30, 29.5, 28.5, 28, 27.5, 27, 27, 26, 26, 25.5, 24, 23.5, 23.5, 22, 21.5, 20, 19, 14.5

Homework 2 due on Monday September 15, 2008

Chapter 1, Problem 2; Chapter 2, Problems 6.

Also:

(A) Let B be a Borel set in $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, where $\mathbb{R} = (-\infty, \infty)$. Prove that each vertical section of B is Borel, that is, for any $x \in \mathbb{R}$ the set $\{y \in \mathbb{R} : (x, y) \in B\} \in \mathcal{B}(\mathbb{R})$.

(B) In the setting of Example 4, Ch 1, of probability space Ω of points $\omega = (\omega_1, \omega_2, \dots)$ define $\varepsilon_j(\omega) = \omega_j$ and take a finite subset $\{k_1, k_2, \dots, k_n\}$ of integers $\{1, 2, 3, \dots\}$ such that $k_1 < k_2 < \dots < k_n$ and take some numbers a_1, a_2, \dots such that $a_j = 0$ or 1. Show that

$$\{\omega : \varepsilon_{k_j}(\omega) = a_j \quad \forall j = 1, \dots, n\}$$

is an event and compute its probability.

(C) In the setting of Example 4, Ch 1, for $\omega = (\omega_1, \omega_2, \dots)$ define

$$X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k}}{2^k}, \quad Y(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k-1}}{2^k}.$$

Show that for any numbers a, b, c, d such that $0 \leq a \leq b \leq 1$ and $0 \leq c \leq d \leq 1$ we have

$$P(\{\omega : (X(\omega), Y(\omega)) \in (a, b] \times (c, d]\}) = (b - a)(d - c).$$

(Hint: Start with $b = d = 1$, use (B) and an argument given in class.)

Ordered results for HW2 (out of 35 pts):

35, 33.5, 32.5, 32, 31, 31, 31, 31, 31, 30, 30, 30, 30, 29.5, 29, 28.5, 28, 28, 27, 27, 26, 25.5, 24, 24, 23.5, 23, 22, 20

Ordered results for HW1+HW2:

66.5, 65.5, 64, 61, 60, 59.5, 59, 59, 59, 58, 58, 58, 57, 57, 57, 56, 56, 55.5, 54.5, 54, 52, 52, 51.5, 49.5, 47.5, 43, 40, 37.5

Homework 3 due on Monday September 29, 2008

Assume the definition of expectation given in class (that is the property asserted in Problem 4, Ch 4). Solve the following problems

Chapter 4, Problems 7, 10, 20, 21, 23, 26, 32. Each problem is worth 5 points.

Doing these problems you may use the version of Proposition 14, page 51, proved in class.

Ordered results for HW3 (out of 35 pts):

35, 35, 35, 35, 34, 34, 34, 33, 33, 33, 33, 32.5, 32, 32, 32, 31.5, 31, 30.5, 30, 30, 29.5, 28.5, 27, 26, 25.5, 25.5, 20.5, 16

Ordered results for HW1+HW2+HW3:

100.5, 97, 97, 95, 94.5, 94, 93, 92, 91.5, 91, 91, 90, 89, 89, 87.5, 87.5, 87, 87, 84, 81, 80.5, 77.5, 77.5, 75, 73, 73, 68.5, 58

Homework 4 due on Monday October 13, 2008

Each problem is worth 5 points.

Chapter 5, Problems 1, 31, 33.

While doing 31 and 33 you need to know the Taylor series for $(1+x)^\alpha$. Also solve the following problems.

A) Prove that the product of two probability generating functions is a probability generating function.

B) Let numbers $a_k^n \geq 0$ and a_k be given for $n, k = 1, 2, \dots$. Assume that $a_k^n \rightarrow a_k$ as $n \rightarrow \infty$ for any k . Also assume $\sum_k a_k^n \rightarrow \sum_k a_k < \infty$. Prove then that

$$\lim_{n \rightarrow \infty} \sum_k |a_k^n - a_k| = 0.$$

C) Let X, X_1, X_2, \dots be a sequence of random variables, $r \in [1, \infty)$. Assume that $\lim_{n \rightarrow \infty} X_n$ exists (a.s.) and equals X (a.s.) Also assume that

$$\lim_{n \rightarrow \infty} E(|X_n|^r) = E(|X|^r) < \infty.$$

Prove that

$$\lim_{n \rightarrow \infty} E(|X_n - X|^r) = 0.$$

(Hint: Case $r = 1$ was done in class. In case $r > 1$ repeat the argument given for $r = 1$ with appropriate modifications.)

D) (Like Problem 36, Ch. 8) Let f be a nonnegative Borel measurable function on $(-\infty, \infty)$, $a, b \in (-\infty, \infty)$, $a \neq 0$. Show that (all integrals are Lebesgue integrals)

$$\int_{-\infty}^{\infty} f(x) dx = |a| \int_{-\infty}^{\infty} f(ax + b) dx.$$

(Hint: Denote $g(x) = f(x)e^{|x|^2}$ and introduce two (finite) measures on Borel subsets of $\mathbb{R} = (-\infty, \infty)$)

$$\mu(B) = \int_{\mathbb{R}} e^{-|x|^2} I_B(x) dx, \quad \nu(B) = |a| \int_{\mathbb{R}} e^{-|ax+b|^2} I_B(ax+b) dx.$$

By using π - and λ -systems or Sierpiński class theorem and the fact that the Riemann integral coincides with the Lebesgue integral on Borel Riemann integrable functions, prove that

$$\int_{\mathbb{R}} e^{-x^2} g(x) dx = |a| \int_{\mathbb{R}} g(ax+b) e^{-|ax+b|^2} dx. \quad)$$

Ordered results for HW4 (out of 35 pts):

35, 35, 34.5, 34.5, 34, 33.5, 33.5, 33, 32, 31.5, 31.5, 31, 30, 29.5, 29.5, 29, 28.5, 28.5, 26, 25.5, 25, 24, 22.5, 22.5, 19.5, 16

Ordered results for HW1+HW2+HW3+HW4:

134.5, 131.5, 130, 130, 129.5, 127.5, 124, 123, 122.5, 122.5, 121, 120, 119.5, 118.5, 117, 116, 115.5, 113, 109.5, 103.5, 100.5, 100, 97, 97, 93.5, 93, 74, 73

Homework 5 due on Monday October 27, 2008

Each problem is worth 5 points.

Chapter 7. Problems 8, 10, 14, 15, 19.

Doing Problem 10 you may use that R extends to a measure on $\sigma(\mathcal{E})$ and some facts from integration theory are already known.

Doing Problem 19 consider the function $f(t) = \mu((0, t])$, $t \geq 0$, and prove that $f(t+s) = f(t) + f(s)$, $t, s \geq 0$.

Also do the following.

A) Let X_1, X_2, \dots be a sequence of nonnegative pairwise uncorrelated random variables such that $\mu_n = EX_n < \infty$ and $\text{Var } X_n \leq M\mu_n$ for all n , where M is a constant. By generalizing an argument given in class, prove that

$$\sum_{n=1}^{\infty} \mu_n = \infty \implies \sum_{n=1}^{\infty} X_n = \infty \quad (\text{a.s.}).$$

B) Recall the following setting given in class. We have a field \mathcal{E} of subset of a set Ω and a finitely additive (finite) nonnegative function R defined on \mathcal{E} . Introduce R^* as an outer measure generated by R :

$$R^*(X) = \inf \left\{ \sum_{n=1}^{\infty} R(A_n) : A_1, A_2, \dots \in \mathcal{E}, \bigcup_n A_n \supset X \right\}.$$

Recall that an $A \subset \Omega$ is called an R^* -set if

$$R^*(X) = R^*(XA) + R^*(XA^c) \quad \forall X \subset \Omega.$$

Also we call an $X \subset \Omega$ a null set if $R^*(X) = 0$.

Prove that null sets are R^* -sets.

Ordered results for HW5 (out of 35 pts):

35, 34.5, 34, 34, 33.5, 33, 33, 33, 33, 32.5, 32, 31.5, 31.5, 31.5, 30.5, 30.5, 28.5, 28.5, 27.5, 27, 27, 25.5, 23.5, 22.5, 22, 17.5, 14.5

Ordered results for HW1+HW2+HW3+HW4+HW5:

167.5, 164.5, 164, 163, 160, 157.5, 157, 156.5, 155, 154.5, 154, 150, 150, 150, 148.5, 147.5, 147.5, 147, 139.5, 129, 125.5, 124.5, 124, 122, 116, 93, 91.5, 87.5

Homework 6 due on Monday November 10, 2008

Each problem is worth 5 points.

Chapter 7. Problems 2, 21 (you are not required to prove countable additivity)

Chapter 8. Problems 1, 3, 6, 7.

Chapter 9. Problem 16.

Ordered results for HW6 (out of 35 pts):

35, 35, 35, 35, 35, 35, 35, 35, 35, 34.5, 34.5, 34.5, 34.5, 34.5, 33, 33, 32.5, 32.5, 32, 31.5, 31, 31, 30.5, 29.5, 29.5, 28, 26, 23.5

Ordered results for HW1+HW2+HW3+HW4+HW5+HW6:

202.5, 199, 196, 195.5, 195, 192.5, 192, 191, 189.5, 189.5, 189, 183, 183, 182, 181, 180.5, 178.5, 177, 171.5, 160, 155, 152.5, 151.5, 149, 147.5, 124, 122

Homework 7 due on Monday November 24, 2008

Each problem is worth 5 points.

Chapter 9. Problem 36, 45, 46.

In Problem 36 prove not what is required but that

$$\lim_{n \rightarrow \infty} \frac{S_n}{M_n} = P(Y \in B) \quad (\text{a.s.}).$$

Chapter 12. Problem 15, 17.

In Problem 17 do not do what is written in the book but only calculate ES_n and find $\lim_{n \rightarrow \infty} S_n^{1/n}$.

Also do the following.

A) Let X_n , $n = 1, 2, \dots$, be pairwise independent random variables such that $P(X_n \in (a, b)) = (1 - 2^{-n})(b - a)$ for $0 \leq a \leq b \leq 1$ and $P(X_n = 2^{-n}) = 2^{-n}$. Show that there exists a constant c such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k = c \quad (\text{a.s.})$$

and find this constant.

B) Let X_1, X_2 be uncorrelated random variables each of which takes only two values. Show that they are independent. (Hint: First assume $X = I_A$ and $Y = I_B$.)

Ordered results for HW7 (out of 35 pts):

35, 35, 35, 35, 34.5, 34.5, 34.5, 34.5, 34, 33, 33, 32.5, 32.5, 32.5, 32.5, 31.5, 30.5, 30, 29.5, 28.5, 28, 27.5, 22.5

Ordered results for HW1+...+HW7:

233, 231.5, 231, 228, 227.5, 227.5, 225, 223.5, 223.5, 222, 219.5, 217.5, 216.5, 215.5, 215.5, 214.5, 212, 211.5, 201.5, 189.5, 182.5, 177, 174, 152.5, 147.5, 124

Take home final. Due before noon Friday December 12 in my office.

Each problem is worth 10 points.

Chapter 1. Problems 2, 7.

Chapter 7. Problem 19.

Chapter 8. Problem 1 (you may prefer not to use the hint).

Chapter 9. Problem 46.

Chapter 12. Problem 15 (m is positive means $m > 0$).

Also do the following.

A) Let X, Y be \mathbb{R}^d -valued random variables each of which takes only countably many values. Assume that

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

for all $x, y \in \mathbb{R}^d$. Prove that X and Y are independent.