

Precise evaluation of the periodized Green’s function for the Helmholtz equation at high frequencies

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Abstract

A difficulty that arises in the context of two-dimensional infinite, d -periodic rough-surface scattering relates to the effective numerical evaluation of the corresponding “quasi-periodic Green function” $G_{qp}(x, y)$. Recently, we introduced a novel scheme, based on the integral representation of $G_{qp}(x, y)$, that can be shown to outperform every alternative numerical evaluation procedure, and is especially effective for high-frequency calculations. In this paper, we extend our algorithm to the evaluation of the partial derivatives of the $G_{qp}(x, y)$ (as necessary, for instance, in the solution of integral equations that involve double layer potential representations). Moreover, we further introduce a stabilizing mechanism based on multi-precision evaluations which, unlike those applicable to more classical algorithms for the calculation of G_{qp} , entails higher precision computation of only a few selected quantities.

1 Introduction

A difficulty that arises in the context of two-dimensional, infinite, d -periodic rough-surface scattering relates to the effective numerical evaluation of the corresponding “quasi-periodic Green function” G_{qp} . Due to its relevance in a variety of applications, this problem has generated significant interest over the last forty years, and a variety of numerical methods have been devised for this purpose. None of these methods to evaluate G_{qp} however, were designed for high-frequency calculations. As a result, in this regime, these methods become prohibitively expensive and/or unstable. In [1, 2] we presented a novel scheme that can be shown to outperform every alternative numerical evaluation procedure, and is especially effective for high-frequency calculations. Our new algorithm is based on the use of some exact integrals that arise on judicious manipulation of the integral representation of G_{qp} and which reduce the overall problem to that of evaluation of a sequence of simpler integrals that can be effectively handled by standard quadrature formulas.

Beyond that of G_{qp} the solution of the scattering problem typically necessitates the evaluation of its normal derivatives (e.g. in connection with integral equations that rely on double layer potential representation). In this paper we show that the derivations in [1, 2] largely extend to this latter case resulting in efficient mechanisms for the numerical evaluation of the partial derivatives of G_{qp} .

In addition to these new developments, we also present here the details of a procedure to further enhance the stability of our new algorithms. As we have shown in [1, 2], and although these schemes allow for efficient and accurate evaluations at unprecedented frequencies, their precision does eventually deteriorate as the frequency increases. We present an analysis that enables the identification of the main source of the resulting inaccuracies and suggests a means for its mollification. As we explain, the problems can be traced back to the significant inaccuracies that can arise on floating point evaluation of trigonometric and special functions for very large values of their arguments which can, of course,

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be substantially reduced by resorting to higher-precision arithmetic. A main point here, relates to the possibility provided by our new schemes which, in contrast with classical approaches, allow for the attainment of very significant accuracy gains through the multi-precision evaluation of only *a few* selected quantities.

The rest of the paper is organized as follows. First in §2 we briefly review the algorithms introduced in [1, 2] and we present their extensions that allow for the evaluation of the partial derivatives of G_{qp} . Then, in §3, we first detail the multi-precision manipulations that lead to enhanced stability characteristics for this class of schemes. Finally, we present a variety of numerical results that confirm that our new procedures allow for calculations of a quality and efficiency comparable to that attainable by state-of-the-art methodologies where these are applicable while also enabling evaluations for frequencies that lie well-beyond those that can be treated with current techniques.

2 The Two-dimensional Periodized Green function And Its Derivatives

A variety of classical representations exist for the evaluation of the quasi-periodic Green function. Notable among these is the so-called spectral representation which, for a fixed incidence angle θ , takes on the form [3]

$$G_{\text{qp}}(x, y) = \frac{1}{2id} \sum_{n=-\infty}^{\infty} \frac{e^{i(\alpha_n x + \beta_n |y|)}}{\beta_n} \quad (1)$$

where k is the wave number, $\alpha = k \sin(\theta)$ and

$$\alpha_n = \alpha + \frac{2\pi n}{d} \text{ and } \beta_n = \sqrt{k^2 - \alpha_n^2}.$$

The series in (1) converges exponentially. However the exponential decay only manifests itself beyond $\pm n = \pm(k \mp \alpha)d/(2\pi)$, and it deteriorates with decreasing $|y|$. As a result, evaluation of G_{qp} at high-frequencies becomes prohibitively expensive using this representation. Similar considerations apply to the myriad of alternative representations that have been derived in the past, such as those based on integral representations [4, 5], on Kummer [6] and Ewald transforms [7] or on lattice sums [8], which are further limited by instabilities that accentuate with increasing frequency. These realizations motivate the need for alternative algorithms that can significantly reduce the computational cost of evaluations for G_{qp} while retaining a stable behavior. Next we review one such procedure which was originally introduced in [1, 2].

2.1 A New Algorithm For The Evaluation Of The Green Function

The method of [1, 2] begins with the integral form of G_{qp} , which is judiciously manipulated to result in the representation

$$G_{\text{qp}}(x, y) = -\frac{i}{4} H_0^{(1)}(kr) - \frac{1}{2\pi} (I^{\text{M}} + S^{\text{M}}) \quad (2)$$

where $r = \sqrt{x^2 + y^2}$ and the integral I^{M} is defined by

$$\begin{aligned} I^{\text{M}} &= \int_0^\infty \left(\frac{e^{iky\sqrt{u^2-2i+kx}(u^2-i)} + e^{-iky\sqrt{u^2-2i+kx}(u^2-i)}}{\sqrt{u^2-2i}(e^{-i\alpha(M+1)d}e^{k(M+1)d(u^2-i)} - e^{-i\alpha M d}e^{kdM(u^2-i)})} \right. \\ &\quad \left. + \frac{e^{iky\sqrt{u^2-2i-kx}(u^2-i)} + e^{-iky\sqrt{u^2-2i-kx}(u^2-i)}}{\sqrt{u^2-2i}(e^{i\alpha(M+1)d}e^{k(M+1)d(u^2-i)} - e^{i\alpha M d}e^{kdM(u^2-i)})} \right) du \\ &\equiv \int_0^\infty f_M(u) du. \end{aligned} \quad (3)$$

The series S^M , on the other hand, is given by

$$S^M = \sum_{j=1}^M A_j^1 \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} v_{n,j}^1 + \sum_{j=1}^M A_j^2 \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} v_{n,j}^2 \quad (4)$$

where the the function $h(u)$ is

$$h(u) = \frac{e^{iky\sqrt{u^2-2i-kyu(1+i)}}}{\sqrt{u^2-2i}} \quad (5)$$

and

$$A_j^a = \sqrt{\frac{\pi}{k(jd+(2a-3)x)}} e^{i\frac{ky^2}{2(jd+(2a-3)x)}} e^{ik(jd+(2a-3)x)} e^{(3-2a)i\alpha jd} \quad (6)$$

for $a = 1, 2$. Finally, the weight's $v_{n,j}^a$ are given recursively by

$$v_{0,j}^a = 1, \quad v_{1,j}^a = \frac{1}{w_j^a} \quad \text{and} \quad v_{n,j}^a = \frac{v_{n-1,j}^a}{w_j^a} + \frac{(n-1)v_{n-2,j}^a}{w_j^a s}$$

with

$$w_j^a = \frac{2(jd+(2a-3)x)}{y(1+i)} \quad \text{and} \quad s = ky(1+i).$$

Clearly large variations of f_M in (3) affect negatively the stability of the evaluations and must be avoided, leading to a choice of

$$M \approx \frac{ky^2}{d}.$$

To complete the prescriptions, we must provide a rule for the truncation of the improper integral defining I^M in (3) and of the infinite inner series in S^M in (4). For a given error ϵ , it can be shown [1, 2] that the integrals in (3) can be truncated to the interval $[0, C]$ where

$$C = \frac{y + \sqrt{y^2 - \frac{4Md \log(\epsilon)}{k}}}{2Md} \sim \frac{1}{ky} \quad \text{and} \quad f^M(C) = \epsilon,$$

since in this case [9, Equations 4.146.1 and 4.146.2]

$$\int_C^\infty f^M(u) du \sim \epsilon \int_0^\infty \frac{e^{(ky-2kd)v+ikyv}}{e^{kd(v^2-i)}} dv \sim \epsilon \left[\sqrt{\frac{\pi}{4kd}} e^{(d-y)/d} e^{iky(y-2d)/(2d)} \right].$$

Also, since the integrand does not oscillate rapidly within the range $[0, C]$, a canonical quadrature can be applied to evaluate it accurately with a computational complexity independent of the wavenumber. Finally, the inner series in S^M must be truncated at a number $n = N$ satisfying [1, 2]

$$N \sim 3 \frac{eky^4}{(2dj)^3}. \quad (7)$$

2.2 Extension To Partial Derivatives of G_{qp}

In addition to precise values of G_{qp} , the solution of scattering problems typically necessitate the evaluation of its normal derivatives (e.g. in connection with integral equations that rely on double layer potential representations). The derivation of the scheme for the effective evaluation of the partial derivatives follows largely along the lines of that for G_{qp} itself. More precisely, we have the representations

$$\begin{aligned} \frac{\partial G_{qp}}{\partial x} &= \frac{ikx}{4r} H_1^{(1)}(kr) - \frac{1}{\pi} \left(\sum_{j=1}^M A_j^1 \sum_{n=0}^{\infty} \frac{g_1^{(n)}(0)}{n!} v_{n,j}^1 + \sum_{j=1}^M A_j^2 \sum_{n=0}^{\infty} \frac{g_1^{(n)}(0)}{n!} v_{n,j}^2 \right. \\ &+ \int_0^\infty \frac{k(u^2-i) [e^{iky\sqrt{u^2-2i+kx(u^2-i)}} + e^{-iky\sqrt{u^2-2i+kx(u^2-i)}}]}{\sqrt{u^2-2i} (e^{-i\alpha(M+1)d} e^{k(M+1)d(u^2-i)} - e^{-i\alpha M d} e^{kdM(u^2-i)})} du \\ &\left. - \int_0^\infty \frac{k(u^2-i) [e^{iky\sqrt{u^2-2i-kx(u^2-i)}} + e^{-iky\sqrt{u^2-2i-kx(u^2-i)}}]}{\sqrt{u^2-2i} (e^{i\alpha(M+1)d} e^{k(M+1)d(u^2-i)} - e^{i\alpha M d} e^{kdM(u^2-i)})} du \right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial G_{\text{qp}}}{\partial y} &= \frac{iky}{4r} H_1^{(1)}(kr) - \frac{1}{\pi} \left(\sum_{j=1}^M A_j^1 \sum_{n=0}^{\infty} \frac{g_2^{(n)}(0)}{n!} v_{n,j}^1 + \sum_{j=1}^M A_j^2 \sum_{n=0}^{\infty} \frac{g_2^{(n)}(0)}{n!} v_{n,j}^2 \right) \\ &+ \int_0^{\infty} iku \frac{e^{iky\sqrt{u^2-2i}+kx(u^2-i)} - e^{-iky\sqrt{u^2-2i}+kx(u^2-i)}}{(e^{-i\alpha(M+1)d} e^{k(M+1)d(u^2-i)} - e^{-i\alpha M d} e^{k d M(u^2-i)})} du \\ &+ \int_0^{\infty} iku \frac{e^{iky\sqrt{u^2-2i}-kx(u^2-i)} - e^{-iky\sqrt{u^2-2i}-kx(u^2-i)}}{(e^{i\alpha(M+1)d} e^{k(M+1)d(u^2-i)} - e^{i\alpha M d} e^{k d M(u^2-i)})} du \end{aligned}$$

where

$$\begin{aligned} g_1(u) &= k(u^2 - i) \frac{e^{iky\sqrt{u^2-2i}-kyu(1+i)}}{\sqrt{u^2 - 2i}} \\ &= k(u^2 - i)h(u) \end{aligned}$$

and

$$\begin{aligned} g_2(u) &= iku e^{iky\sqrt{u^2-2i}-kyu(1+i)} \\ &= iku \sqrt{u^2 - 2i} h(u) \end{aligned}$$

and where the sums and integrals can be truncated exactly as those corresponding to G_{qp} .

3 Numerical Results

In this section, we present a variety of numerical results that demonstrate the extended applicability and efficiency of our new schemes when compared to alternative methods. As we have anticipated, an additional advantage of our new procedures relates to their amenability to simple “multi-precision interactions” that allow for the evaluation of a small number of quantities in higher precision to further significantly improve on the accuracy and applicability of the approach. The details on these interactions are presented next in §3.1 and the results of some representative numerical experiments follow in §3.2.

3.1 Multi-precision Interaction

A difficulty that arises for every method that computes G_{qp} relates to the evaluation in finite precision arithmetic of highly oscillatory functions as the frequency increases, due to the large relative errors in their values that might result from small relative errors in their phases. For instance, the evaluation of the quantity

$$E = e^{i2\pi x} \quad (8)$$

for large values of x becomes inexact as the calculation of

$$x_{\text{mp}} = \text{mod}(2\pi x, 2\pi) \quad (9)$$

gets progressively more inaccurate with increasing x . Indeed, for finite precision arithmetic

$$x_{\text{mp}}^{\text{evaluated}} = x_{\text{mp}}^{\text{exact}} + \epsilon \quad (10)$$

where the error is

$$\epsilon \approx \mathcal{O}(x10^{-\text{precision}}). \quad (11)$$

From (10) then we have

$$\begin{aligned} \cos^{\text{exact}}(2\pi x) - \cos^{\text{evaluated}}(2\pi x) &= \cos(x_{\text{mp}}^{\text{exact}}) - \cos(x_{\text{mp}}^{\text{evaluated}}) \\ &= -2 \sin(x_{\text{mp}}^{\text{exact}} + x_{\text{mp}}^{\text{evaluated}}) \sin(\pi\epsilon) + \mathcal{O}(10^{-\text{precision}}) \\ &\approx \sin(x_{\text{mp}}^{\text{exact}} + x_{\text{mp}}^{\text{evaluated}}) \mathcal{O}(\epsilon) + \mathcal{O}(10^{-\text{precision}}) \text{ for } \epsilon \ll 1, \end{aligned} \quad (12)$$

and

$$\begin{aligned}\sin^{\text{exact}}(2\pi x) - \sin^{\text{evaluated}}(2\pi x) &= \sin(x_{\text{mp}}^{\text{exact}}) - \sin(x_{\text{mp}}^{\text{evaluated}}) \\ &= 2 \cos\left(\frac{x_{\text{mp}}^{\text{exact}} + x_{\text{mp}}^{\text{evaluated}}}{2}\right) \sin(\pi\epsilon) + \mathcal{O}(10^{-\text{precision}}) \\ &\approx \cos\left(\frac{x_{\text{mp}}^{\text{exact}} + x_{\text{mp}}^{\text{evaluated}}}{2}\right) \mathcal{O}(\epsilon) + \mathcal{O}(10^{-\text{precision}}) \text{ for } \epsilon \ll 1\end{aligned}$$

which imply that the evaluation of the quantity in (8) in finite precision will generically result in a loss of $\log_{10}(|x|)$ significant digits. An example of this, for the value $x = 10^4$, is presented in Table 1. We note here that, according to (12), the real part of E in this case is computed exactly, as x being an integer leads to a round-off error that is, in fact, proportional to $\epsilon^2 \ll 10^{-\text{precision}}$.

Table 1: Imaginary part of the complex quantity $e^{i(2\pi)10^4}$.

Method	$\text{Im}(e^{i(2\pi)10^4})$
Euler's Formula	0
Double precision	-9.713647079136980e-13
Multi precision with 48 digit accuracy	-0.978835805011081538436718742758200274... e-43i

Clearly, from (11) the error can be reduced by simply computing

$$x_{\text{mp}}^{\text{new}} = \text{mod}(x, 1). \quad (13)$$

with higher accuracy and evaluating (with the underlying floating point precision)

$$\begin{aligned}E &= e^{i2\pi x} \\ &= e^{i2\pi x_{\text{mp}}^{\text{new}}}\end{aligned}$$

(Note that if x is an integer (13) can be readily calculated exactly in integer arithmetic.)

In connection with these considerations, and as we have stated, our schemes display an additional advantage over alternative procedures as they allow for enhanced accuracy through the multi-precision evaluation of *only* a few quantities with a number of additional digits that grows only logarithmically with frequency. More precisely, for this class algorithms, effective multi-precision interactions can be accomplished through three successive steps, namely:

Step 1

It can be shown that the dominant instability in formula (2) arises from the evaluations of the quantities

$$B_j = e^{ijkd} \quad (14)$$

for large values of k . When $\frac{d}{2\pi}k$ is not an integer, the quantity

$$k_{\text{mp}} = \text{mod}\left(\frac{d}{2\pi}k, 1\right) \quad (15)$$

is evaluated in multi-precision and inserted into (3) and (6) when computing the value of

$$B_j = e^{i2\pi j k_{\text{mp}}}$$

in (14).

In the experiments that follow

$$k = 10^n + 0.2 \text{ for } n \in N \quad (16)$$

and the period is taken to be $d = 2\pi$ so that

$$k_{\text{mp}} = \text{mod}\left(\frac{d}{2\pi}k, 1\right) = 0.2$$

can be enforced readily within the underlying precision.

Step 2

Beyond the possible inaccuracies that arise as a result of the problem described in Step 1, the next order error in floating point evaluations of (2) arises from the calculation of $H_0^{(1)}(kr)$, $e^{\pm i\alpha jd}$ and $e^{\pm ikx}$. Similarly to Step 1, these values then are calculated accurately by evaluating $H_0^{(1)}(kr)$, $\text{mod}(\frac{d}{2\pi}\alpha, 1)$ and $\text{mod}(\frac{kx}{2\pi}, 1)$ in higher precision.

Step 3

Finally, with Steps 1 and Step 2 enforced, the remaining error arises mainly from the evaluation of the quantity

$$C_j^a = e^{i\frac{ky^2}{2j(d+(2a-3)x)}} \quad (17)$$

for $a = 1, 2$ that appears in A_j^a defined in (6). To detail our approach to the evaluation of (17) it will be convenient to distinguish two cases, namely

1. $x = 0$: In this case we have $C_j^1 = C_j^2 = e^{i(\frac{ky^2}{4\pi jd})2\pi}$ and we begin by letting

$$\frac{ky^2}{4\pi d} = r_{\text{mp}} + n$$

where the simple quantity

$$r_{\text{mp}} = \text{mod}\left(\frac{ky^2}{4\pi d}, 1\right)$$

is evaluated in multi-precision and inserted in (17). More precisely, we compute

$$\frac{ky^2}{4dj\pi} = \frac{r_{\text{mp}}}{j} + \frac{n_1}{j} + \frac{n_2}{j}, \quad C_j^1 = C_j^2 = e^{i\frac{(r_{\text{mp}}+n_1)}{j}2\pi} \quad (18)$$

where $n_1 = \text{mod}(n, j)$ and $n_2 = n - \text{mod}(n, j)$ and, importantly, the latter evaluations of the “mod” function are performed in *integer arithmetic*.

2. Arbitrary x : For the general case, we can write

$$\frac{ky^2}{2} \frac{1}{(jd + (2a - 3)x)} = \frac{ky^2}{2} \left[\sum_{n=1}^N \left(\frac{1}{jd} \right)^n (-2a - 3)x^{n-1} + \frac{(-(2a - 3)x)^N}{(jd)^N (jd + (2a - 3)x)} \right]$$

for $a = 1, 2$ and, consequently,

$$\begin{aligned} C_j^a &= e^{i\frac{ky^2}{2(jd+(2a-3)x)}} \\ &= \left(\prod_{n=1}^N e^{i\frac{ky^2(-(2a-3)x)^{n-1}}{2(jd)^n}} \right) e^{i\frac{ky^2(-(2a-3)x)^N}{2(jd)^N(jd+(2a-3)x)}}. \end{aligned} \quad (19)$$

The product of the first N factors in the right-hand side of (19) can be computed as in 1. above. The last factor, in turn, can be calculated directly provided N is chosen so that $\frac{ky^2 x^N}{2d^N(d-x)} \sim \mathcal{O}(1)$.

3.2 Numerical Examples

In this last section we provide a number of numerical tests that exemplify the developments in §§2.2 and 3.1. First, in Tables 2–3, we illustrate the effect of the multi-precision interactions described in §3.1 for a configuration with normal incidence ($\alpha = 0$) and $k = 10^8 + 0.2$ and for a case when $x = 0$ and one where $x \neq 0$ (cf. (18), (19)). We see that, in every case, each step described in §3.1 does indeed result in substantive accuracy gains.

Table 2: Real and imaginary parts of $G_{\text{qp}}(x, y)$ with multi-precision interference for $\alpha = 0$, $k = 10^8 + 0.2$, $y = 0.1$ and $x = 0$.

Method	$\text{Re}(G_{\text{qp}}(x, y))$	$\text{Im}(G_{\text{qp}}(x, y))$
Exact	$4.020872995782040e - 05$	$5.290100535833349e - 05$
G_{qp}	<u>$4.020891789883519e - 05$</u>	<u>$5.290082485712432e - 05$</u>
G_{qp} with correct k_{mp}	<u>$4.020872992531304e - 05$</u>	<u>$5.290100544159487e - 05$</u>
G_{qp} with correct $H_0^{(1)}(kr)$	<u>$4.020872995771583e - 05$</u>	<u>$5.290100535838623e - 05$</u>
G_{qp} with correct r_{mp}	<u>$4.020872995782107e - 05$</u>	<u>$5.290100535833215e - 05$</u>

Table 3: Real and imaginary parts of $G_{\text{qp}}(x, y)$ with multi-precision interference for $\alpha = 0$, $k = 10^8 + 0.2$, $y = 2k^{-1/4}$ and $x = 0.1$.

Method	$\text{Re}(G_{\text{qp}}(x, y))$	$\text{Im}(G_{\text{qp}}(x, y))$
Exact	$6.1609746496066595e - 05$	$-1.6683169948859071e - 05$
G_{qp}	<u>$6.160958396546703e - 05$</u>	<u>$-1.668337250974780e - 05$</u>
G_{qp} with correct k_{mp}	<u>$6.160974652968465e - 05$</u>	<u>$-1.668316991650857e - 05$</u>
G_{qp} with correct $H_0^{(1)}(kr)$	<u>$6.160974651343166e - 05$</u>	<u>$-1.668316996572452e - 05$</u>
G_{qp} with correct $\text{mod}(\frac{kx}{2\pi}, 1)$	<u>$6.160974649606982e - 05$</u>	<u>$-1.668316994885799e - 05$</u>
G_{qp} with correct $r_{\text{mp}} (N = 1)$	<u>$6.160974649606636e - 05$</u>	<u>$-1.668316994885899e - 05$</u>

The second set of results, in Tables 4– 5, show experiments related to the evaluation of the partial derivatives of $G_{\text{pq}}(x, y)$ (cf. §2.2) for $(x, y) = (0.1, 0.1)$ and wavenumbers as in (16) with $3 \leq n \leq 8$. In these tables we compare the accuracy and computational times (t) that result from an implementation of the new algorithm (labeled “NA”) in §2.2 and that of some alternative schemes, including the classical spectral representation (“Spe.”), a method based on Ewald’s transformation (“Ewa.”) and the lattice sums method (“LSM”); several other methods (spatial sums, Kummer transform, integral representations [1]) are excluded due to their very poor convergence properties. (Also, we note that the LSM times correspond solely to the *evaluation* of the relevant sums and they *do not* include the time for the generation of the coefficients. The latter times become inordinate for large values of k and thus prevent the true evaluation; in tables 4– 5, we therefore simply write “...” for the values, and we include estimated times garnered from evaluation of the corresponding series using arbitrary, random coefficients).

For the evaluation of the relative error

$$\text{err}_x = \frac{\left| \frac{\partial G_{\text{qp}}(k, x, y)^{\text{Exact}}}{\partial x} - \frac{\partial G_{\text{qp}}(k, x, y)^{\text{Method}}}{\partial x} \right|}{\left| \frac{\partial G_{\text{qp}}(k, x, y)^{\text{Exact}}}{\partial x} \right|}, \quad (20)$$

and

$$\text{err}_y = \frac{\left| \frac{\partial G_{\text{qp}}(k, x, y)^{\text{Exact}}}{\partial y} - \frac{\partial G_{\text{qp}}(k, x, y)^{\text{Method}}}{\partial y} \right|}{\left| \frac{\partial G_{\text{qp}}(k, x, y)^{\text{Exact}}}{\partial y} \right|} \quad (21)$$

an “exact solution” was computed in quadruple precision arithmetic using the spectral representation. These results clearly demonstrate the significant savings in computational cost that can be attained with the use of the new procedures, without compromising accuracy.

Table 4: Error (20) and computational times (t) for evaluation of $\frac{\partial G_{qp}}{\partial x}$ with $\alpha = 0, k = 10^n + 0.2, (x, y) = (0.1, 0.1)$.

n	Spe.	t	Ewa.	t	LSM.	t	NA	t
3	3.9e-14	0.02s	3.9e-14	0.4s	7.1e-12	0.04s	3.7e-14	0.04s
4	1.0e-11	0.04s	1.0e-11	1s	8.0e-12	0.2s	3.6e-13	0.04s
5	5.6e-11	0.2s	5.6e-11	7s	8.9e-11	1.4s	1.6e-12	0.04s
6	2.1e-10	1.6s	2.1e-10	64s	...	14s	1.3e-11	0.04s
7	2.4e-07	17s	2.4e-07	670s	...	141s	6.9e-10	0.08s
8	3.7e-07	280s	3.7e-07	104m	...	1414s	4.5e-10	0.4s

Table 5: Error (21) and computational times (t) for evaluation of $\frac{\partial G_{qp}}{\partial y}$ with $\alpha = 0, k = 10^n + 0.2, (x, y) = (0.1, 0.1)$.

n	Spe.	t	Ewa.	t	LSM.	t	NA	t
3	1.0e-14	0.02s	1.0e-14	0.4s	7.8e-14	0.04s	2.9e-14	0.04s
4	2.6e-13	0.04s	2.6e-13	1s	2.3e-13	0.2s	2.7e-13	0.04s
5	1.6e-13	0.2s	1.6e-13	7s	2.9e-12	1.4s	1.6e-12	0.04s
6	3.0e-12	1.6s	3.0e-12	64s	...	14s	1.4e-11	0.04s
7	8.2e-12	17s	8.2e-12	670s	...	142s	1.4e-10	0.09s
8	3.8e-10	280s	3.8e-10	104m	...	1412s	2.3e-09	0.4s

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