COHOMOLOGY OF AUTOMORPHIC BUNDLES

KAI-WEN LAN

ABSTRACT. In this survey article, we review some recent works (by the author and his collaborators Junecue Suh, Michael Harris, Richard Taylor, Jack Thorne, and Benoît Stroh) on the cohomology of automorphic bundles over locally symmetric varieties and some related geometric objects.

1. Classical story: modular curves and forms

Let us begin by briefly reviewing the classical story of modular curves and modular forms. For more details and references, we shall refer the readers to the first two sections of the survey article [Lan12b], which had a similar starting point. Nevertheless, our goals in this article are more general, with less emphasis on the good reduction integral models of PEL-type Shimura varieties.
1.1. Classical modular forms. The group $\text{SL}_2(\mathbb{R})$ acts on the Poincaré upper-half plane

$$\mathcal{H} = \{ z \in \mathbb{C} : \text{im}(z) > 0 \}$$

by the usual Möbius transformation

$$\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R}) : z \mapsto \gamma(z) := \frac{az + b}{cz + d}. \tag{1.1}$$

This is induced by the (transitive) action of $\text{SL}_2(\mathbb{C})$ on the projective coordinates of $\mathbb{P}^1(\mathbb{C})$:

$$\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{C}) : \left( \begin{array}{c} z \\ 1 \end{array} \right) \mapsto \left( \begin{array}{c} az + b \\ cz + d \end{array} \right) \sim \left( \begin{array}{c} \gamma(z) \\ 1 \end{array} \right). \tag{1.2}$$

Note that $\sim$ is given by division by the factor $(cz + d)$.

Suppose $\Gamma \in \text{SL}_2(\mathbb{Z})$ is a finite index subgroup (which is then an arithmetic subgroup of $\text{SL}_2(\mathbb{Q})$). Suppose $k \in \mathbb{Z}$ is an integer.

**Definition 1.1.** A classical modular form of level $\Gamma$ and weight $k$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying the following two conditions:

1. **(automorphy condition)** $f(\gamma(z)) = (cz + d)^k f(z)$ for all $\gamma \in \Gamma$.
2. **(growth condition)** $(cz + d)^{-k} f(\gamma(z))$ stays bounded as $\text{im}(z) \to +\infty$, for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

We say that $f$ is a cusp form if, instead of (2), it satisfies the following:

3. **(cuspidal condition)** $(cz + d)^{-k} f(\gamma(z)) \to 0$ as $\text{im}(z) \to +\infty$, for all $\gamma \in \text{SL}_2(\mathbb{Z})$.

We shall denote the space of modular forms (resp. cusp forms) of level $\Gamma$ and weight $k$ by $M_k(\Gamma; \mathbb{C})$ (resp. $S_k(\Gamma; \mathbb{C})$). (These might be zero spaces.)

1.2. Modular curves and a geometric definition of modular forms. We can redefine $M_k(\Gamma; \mathbb{C})$ and $S_k(\Gamma; \mathbb{C})$ geometrically. The group $\Gamma$ acts naturally on $\mathcal{H}$ and on $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, and we have the open and compactified modular curves

$$Y_\Gamma := \Gamma \backslash \mathcal{H} \hookrightarrow X_\Gamma := \Gamma \backslash \mathcal{H}^* \tag{1.3}$$

(with a suitable topology on $\mathcal{H}^*$). The pullback of $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(1)$ to $\mathcal{H}$ descends to a line bundle $\omega$ on $Y_\Gamma$, which extends to a line bundle $\omega$ on $X_\Gamma$ such that

$$M_k(\Gamma; \mathbb{C}) \cong H^0(X_\Gamma, \omega^\otimes k) \tag{1.4}$$

and

$$S_k(\Gamma; \mathbb{C}) \cong H^0(X_\Gamma, \omega^\otimes (-\infty)), \tag{1.5}$$

where $(-\infty)$ means vanishing at the cusps $X_\Gamma - Y_\Gamma = \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$.

By Serre duality and the Kodaira–Spencer isomorphism $\omega^2(-\infty) \cong \Omega^1_{X_\Gamma/\mathbb{C}}$, we have $S_k(\Gamma; \mathbb{C}) \cong H^0(X_\Gamma, \omega^\otimes (-\infty)) \cong H^1(X_\Gamma, \omega^\otimes (2-k))$, where the complex conjugations (denoted by overlines) are induced by integrations on modular curves. When $k \geq 2$, we have the Eichler–Shimura isomorphism

$$H^1(Y_\Gamma, \text{Sym}^{k-2}(\mathbb{C}^{\otimes 2})) \cong M_k(\Gamma; \mathbb{C}) \oplus S_k(\Gamma; \mathbb{C}),$$

which can be rewritten as

$$H^1(Y_\Gamma, \text{Sym}^{k-2}(\mathbb{C}^{\otimes 2})) \cong H^0(X_\Gamma, \omega^\otimes k) \oplus H^1(X_\Gamma, \omega^\otimes (2-k)). \tag{1.5}$$
When \( k = 2 \), this is a consequence of the degeneration of Hodge spectral sequence:

\[
H^1(Y_{\Gamma}, C) \cong H^0(X_{\Gamma}, \Omega^1_{X_{\Gamma}/C}(\log \infty)) \oplus H^1(X_{\Gamma}, \mathcal{O}_{X_{\Gamma}}).
\]

The previous isomorphism (1.5) for general \( k > 2 \) is a similar consequence for nontrivial coefficient systems (using mixed Hodge theory).

### 1.3. Integral models and some applications.

In this subsection, let us assume that \( \Gamma \) is a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \), which means \( \Gamma \) contains the principal congruence subgroup \( \Gamma(N) := \ker(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})) \) for some integer \( N \geq 1 \).

For any rational prime \( p \) and any field isomorphism \( C \cong \mathbb{Q}_p \), we can compare \( H^1(Y_{\Gamma}, \text{Sym}^{k-2}(C^\oplus 2)) \) with \( H^1_{\acute{e}t}(Y_{\Gamma}, \text{Sym}^{k-2}(\mathbb{Q}_p^\oplus 2)) \), and such a comparison is compatible with varying \( \Gamma \) (and with the Hecke actions on these spaces). An important fact is that \( Y_{\Gamma} \) admits a model over some number field \( K \), and hence \( \text{Gal}(\overline{K}/K) \) acts on \( H^1_{\acute{e}t}(Y_{\Gamma}, \text{Sym}^{k-2}(\mathbb{Q}_p^\oplus 2)) \). Very roughly speaking (we are being intentionally vague here), this provides the main source of Galois representations attached to cusp forms of weight \( k \geq 2 \). By using also some integral model of \( X_{\Gamma} \) (over the integers \( \mathcal{O}_K \)), we obtain information about the restriction of such Galois representations to \( \text{Gal}(\mathbb{K}_v/K_v) \) at non-Archimedean places \( v \). By using integral models differently, we can define the integral versions \( M_k(\Gamma; \mathcal{O}_K) \) and \( S_k(\Gamma; \mathcal{O}_K) \), and study congruences between modular forms. Then we can extend the attachment of Galois representations to the case of the low weight \( k = 1 \) (see [DS74]).

In what follows, our goals will be to explain the generalizations of modular curves and modular forms to higher dimensions, and to provide a survey of some recent results due to this author (and his collaborators Junecue Suh, Michael Harris, Richard Taylor, Jack Thorne, and Benoît Stroh) about such generalizations.

## 2. Locally Symmetric Varieties and Automorphic Bundles

Here is an overview of the generalizations we shall explain:

<table>
<thead>
<tr>
<th>classical story</th>
<th>generalizations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{SL}_2 ) or ( \text{GL}_2 )</td>
<td>more general algebraic groups</td>
</tr>
<tr>
<td>( \Gamma )</td>
<td>arithmetic subgroups</td>
</tr>
<tr>
<td>( \text{Poincaré upper-half plane} \ \mathcal{H} )</td>
<td>Hermitian symmetric domains</td>
</tr>
<tr>
<td>open modular curve ( \Gamma \backslash \mathcal{H} )</td>
<td>locally symmetric varieties or Shimura varieties</td>
</tr>
<tr>
<td>compactified modular curve ( \Gamma \backslash \mathcal{H}^* )</td>
<td>various compactifications</td>
</tr>
<tr>
<td>( \omega^k ) and ( \text{Sym}^k(\mathbb{C}^\oplus 2) )</td>
<td>automorphic (vector) bundles (two kinds)</td>
</tr>
<tr>
<td>( M_k = \text{sections of } \omega^k )</td>
<td>sections and also cohomology of automorphic bundles</td>
</tr>
<tr>
<td>Eichler–Shimura isomorphism</td>
<td>Faltings’s dual BGG spectral sequence (degeneration by mixed Hodge theory)</td>
</tr>
<tr>
<td>integral models</td>
<td>integral models</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
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### 2.1. Locally symmetric varieties and their compactifications.

Let us start with some generalization of the Poincaré upper-half plane. For simplicity of exposition, consider the following setup:
• $\mathcal{H}$ is Hermitian symmetric domain of dimension $d$, and
• $G$ is a simply-connected connected semisimple algebraic group over $\mathbb{Q}$ such that $\mathcal{H} \cong G(\mathbb{R})/K$ for some maximal compact subgroup $K$ of $G$. (The setup here can be generalized to the case where $\mathcal{H}$ is a finite disjoint union of Hermitian symmetric domains with a transitive action of $G(\mathbb{R})$, where $G$ is a possibly disconnected reductive algebraic group satisfying some conditions. This more general setup will be tacitly allowed when we discuss about Shimura varieties and their integral models.) Then there is the Borel embedding
\[ \mathcal{H} = G(\mathbb{R})/K \hookrightarrow \mathcal{H}^\vee = G(\mathbb{C})/P(\mathbb{C}) \]
for some maximal parabolic subgroup $P$ of $G_C$, with a Levi subgroup $M$ such that $M(\mathbb{C}) = K_C$ in $G(\mathbb{C})$. (See, e.g., [Hel01, Ch. III, Sec. 7], [AMRT75, Ch. III, Sec. 2.1], and [Mil90, Sec. III.1].) This generalizes the canonical embedding of the Poincaré upper-half plane into the projective line $\mathbb{P}^1(\mathbb{C})$.

Let $\Gamma$ be any arithmetic subgroup of $G(\mathbb{Q})$ which we assume to be neat (so that it acts freely on $\mathcal{H}$), and let
\[ X := \Gamma \backslash \mathcal{H}. \]
Then we have some useful compactifications of $X$, generalizing (1.2) in two ways:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{can}} & X^\text{tor} \\
\downarrow & & \downarrow \\
X^\text{min} & \xrightarrow{\text{can}} & X^\text{tor}
\end{array}
\]

In this diagram, the minimal compactification $X^\text{min}$ due to Satake and Baily–Borel (see [BB66]) is canonical, normal, and projective. This shows that $X$ is quasi-projective and canonically algebraic. But the generally rather singular $X^\text{min}$ is not a good starting point for generalizing modular forms. On the other hand, the toroidal compactifications $X^\text{tor}$ due to Mumford and others (see [AMRT75] and [AMRT10]) are noncanonical, but still canonically depend on certain cone decompositions (or fans), and can be chosen to be projective and smooth with boundaries given by simple normal crossings divisors. They are useful for applications of mixed Hodge theory, and for defining and studying generalizations of modular forms.

2.2. Automorphic bundles and canonical extensions. To define the desired generalizations of modular forms, we need the automorphic (vector) bundles over $X$, together with their canonical and subcanonical extensions over $X^\text{tor}$.

Let $\text{Rep}_C(G_C)$ (resp. $\text{Rep}_C(P)$, resp. $\text{Rep}_C(M)$) denote the category of finite dimensional representations of $G_C$ (resp. $P$, resp. $M$) over $\mathbb{C}$. Given any object $W$ in $\text{Rep}_C(P)$, we can naturally define a vector bundle
\[ W := (G(\mathbb{C}) \times W)/P(\mathbb{C}) \]
over $\mathcal{H}^\vee = G(\mathbb{C})/P(\mathbb{C})$, which pulls back to $\mathcal{H}$ and descends to $X = \Gamma \backslash \mathcal{H}$, still denoted by $W$. By abuse of notation, we shall denote the corresponding sheaf of sections over $X$ by the same symbol $W$. Given any object $W$ in $\text{Rep}_C(M)$, we can pull it back to an object $W$ in $\text{Rep}_C(P)$ via the canonical homomorphism $P \rightarrow M$ and define a vector bundle $W'$ (and its sheaf of sections) over $X$. On the other hand, given any object $V$ in $\overline{\text{Rep}_C(G_C)}$, its restriction $V|_P$ defines a vector bundle
\[ V := (V|_p) \text{ over } X, \text{ and the Lie } G_C \text{ action on } V \text{ defines an integrable connection} \]
\[ \nabla : V \to V \otimes_{\mathcal{O}_X} \Omega^1_{X/C}. \]
Such \( W \) and \((V, \nabla)\) will be called automorphic bundles over \( X \).

According to Mumford and Harris (see Mum77 and Har89), we have a canonical extension
\[ W^\text{can} \]
of \( W \) to a vector bundle over \( X^\text{tor} \), for each \( W \) as above. As explained in Har89 Sec. 4, Deligne’s canonical extension \((V^\text{can}, \nabla^\text{can})\) of \((V, \nabla)\) over \( X^\text{tor} \) (see Del70) is compatible with the above in the sense that \( V^\text{can} = (V|_p)^\text{can} \). Following Har90 Sec. 2, we introduce the subcanonical extensions
\[ W^\text{sub} := W^\text{can}(-D) \]
and
\[ V^\text{sub} := V^\text{can}(-D) \]
of \( W \) and \( V \) over \( X^\text{tor} \), respectively, where
\[ D := (X^\text{tor} - X)_{\text{red}} \]
is the boundary divisor. Then we obtain integrable connections
\[ \nabla : V^\text{can} \to V^\text{can} \otimes_{\mathcal{O}_X} \Omega^1_{X^\text{tor}/C}(\log D) \]
and
\[ \nabla : V^\text{sub} \to V^\text{sub} \otimes_{\mathcal{O}_X} \Omega^1_{X^\text{tor}/C}(\log D) \]
with log poles along \( D \), extending (2.1), for each \( V \) as above.

The coherent sheaf cohomology groups \( H^i(X^\text{tor}, W^\text{can}) \) and \( H^i(X^\text{tor}, W^\text{sub}) \) are called the coherent cohomology of \( X^\text{tor} \) of weight \( W \), which are natural generalizations of modular forms (cf. (1.3) and (1.4)). The hypercohomology of the de Rham complex of \( V^\text{can} \) (resp. \( V^\text{sub} \)), defined by the connection above with log poles, compute the de Rham cohomology \( H^*_\text{dR}(X, V) \) (resp. the compactly supported \( H^*_{\text{dR},c}(X, V) \)).

2.3. Cohomological weights and dual BGG decompositions. Let us compatibly choose weights \( X_{G_C} = X_M \) and positive roots \( \Phi^+_C \) and \( \Phi^+_M \). Then \( \Phi_C^+ \supset \Phi_M^+ \) and \( X_C^+ \subset X_M^+ \). For each \( \mu \in X_{G_C}^+ \) and \( \nu \in X_M^+ \), we shall denote by \( V_{\mu} \in \text{Rep}_C(G_C) \) and \( W_{\nu} \in \text{Rep}_C(M) \) the irreducible representations of highest weight \( \mu \) and \( \nu \), respectively. For simplicity, we shall make various minor adjustments in the notation and terminologies without explicitly introducing them. For example, we shall say that \( H^i(X^\text{tor}, W^\text{can}) \) and \( H^i(X^\text{tor}, W^\text{sub}) \) are coherent cohomology of weight \( \nu \), rather than of weight \( W_\nu \).

Consider \( \rho = \rho_{G_C} := \frac{1}{2} \sum_{\alpha \in \Phi_C^+} \alpha, \rho_M := \frac{1}{2} \sum_{\alpha \in \Phi_M^+} \alpha, \text{ and } \rho^M := \rho - \rho_M \).

Consider the Weyl groups \( W = W_{G_C} \supset W_M, \text{ and consider the minimal length representatives } W^M := \{ w \in W : w(X_C^+) \subset X_M^+ \} \text{ of } W_M \backslash W \). Consider the usual dot action of \( W \) on \( X \) given by \( \cdot w = w(\mu + \rho) - \rho \).

**Definition 2.2.** We say \( \nu \in X_M^+ \) is (de Rham) cohomological if there exist \( \mu = \mu(\nu) \in X_{G_C}^+ \) and \( w = w(\nu) \in W^M \) such that \( W_{\nu} \cong W^\nu_{\cdot w, \mu} \).
This notion is justified by Faltings’s dual BGG spectral sequence (see [Fal83, Sec. 3 and 7] and [FC90, Ch. VI, Sec. 5]) and its degeneracy due to the theory of mixed Hodge modules (see [Sai90] and [HZ01, Cor. 4.2.3]):

$$H^i_{dR}(X, V^{\vee}_\mu) \cong \bigoplus_{w \in W^M} H^{i - l(w)}(X^\text{tor}, (W^\text{can}_{w, \mu})^{\vee})$$ (2.3)

and

$$H^i_{dR,c}(X, V^{\vee}_\mu) \cong \bigoplus_{w \in W^M} H^{i - l(w)}(X^\text{tor}, (W^\text{sub}_{w, \mu})^{\vee})$$ (2.4)

We shall call such isomorphisms dual BGG decompositions. The de Rham cohomology at the left-hand sides of (2.3) and (2.4) can be compared with the Betti (singular) and étale cohomology with the coefficients in some corresponding sheaves, while the coherent cohomology on the right-hand sides of (2.3) and (2.4) are generalizations of modular forms. But only coherent cohomology of cohomological weights can contribute to the de Rham cohomology via the dual BGG decompositions.

**Example 2.5 (G = SL_2; modular curve case).**

$$\mu = k, \quad W_\nu \cong W^{\vee}_{w, \mu}$$

$$l(w) = ? \quad \nu = ?$$

<table>
<thead>
<tr>
<th>l(w) = 0</th>
<th>l(w) = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

In this case, since $V_k = V_k^{\vee} = \text{Sym}^k(C^{\otimes 2})$ for $k \geq 0$ and $W^\text{can}_k = (W^\text{can}_{-k})^{\vee} = \omega^k$ for all $k$, the dual BGG decomposition (2.3) can be identified with the Eichler–Shimura isomorphism (1.5) (for weights $\geq 2$ or $\leq 0$):

$$H^1_{dR}(X, V_k) \cong H^0(X^\text{tor}, W^\text{can}_{k+2}) \oplus H^1(X^\text{tor}, W^\text{can}_{-k}).$$

Note that weight 1 modular forms do not contribute to the de Rham cohomology (but can be studied by congruences with forms of cohomological weights).
**Example 2.6** (G = SL$_2,F$ with $F/Q$ real quadratic; Hilbert modular surfaces).

\[
\mu = (k_1, k_2), \quad W_\nu \cong W^\vee_{\nu,w,\mu} \\
l(w) = ? \\
\nu = ?
\]

\[
l(w) = 1 \\
(\begin{array}{c}
(1, 0) \\
(1, -1)
\end{array})
\]

\[
l(w) = 2 \\
(\begin{array}{c}
(1, 0) \\
(2, 0)
\end{array})
\]

\[
l(w) = 3 \\
(\begin{array}{c}
(1, 0) \\
(3, 0)
\end{array})
\]

In this case, the dual BGG decomposition gives

\[
H^2_{dR}(X, W_{(k_1, k_2)}) \cong H^0(X^{tor}, W_{(k_1+2, k_2+2)}^{can}) \oplus H^1(X^{tor}, W_{(-k_1, k_2+2)}^{can}) \\
\oplus H^2(X^{tor}, W_{(-k_1, -k_2)}^{can})
\]

**Example 2.7** (G = Sp$_4$; Siegel threefolds).

\[
\mu = (k_1, k_2), \quad W_\nu \cong W^\vee_{\nu,w,\mu} \\
l(w) = ? \\
\nu = ?
\]

\[
l(w) = 1 \\
(\begin{array}{c}
(2, 0) \\
(1, 0)
\end{array})
\]

\[
l(w) = 2 \\
(\begin{array}{c}
(3, 1) \\
(2, 0)
\end{array})
\]

\[
l(w) = 3 \\
(\begin{array}{c}
(3, 1) \\
(3, 0)
\end{array})
\]

In this case, the dual BGG decomposition gives

\[
H^3_{dR}(X, W_{(k_1, k_2)}) \cong H^0(X^{tor}, W_{(k_1+3, k_2+3)}^{can}) \oplus H^1(X^{tor}, W_{(-k_1+3, -k_2+1)}^{can}) \\
\oplus H^2(X^{tor}, W_{(-k_2, -k_1)}^{can}) \oplus H^3(X^{tor}, W_{(-k_2, -k_1)}^{can})
\]
3. Vanishing theorems

In this section, we present some vanishing theorems for the cohomology of automorphic bundles over general locally symmetric varieties, in the author’s collaborations with Junecue Suh and Benoît Stroh, together with some examples.

3.1. Positivity of automorphic line bundles. By [BB66], the canonical bundle $W_\mathbb{Z}^{\rho^m} \cong \Omega^d_{X^\mathbb{C}}$ of $X$, where $d = \dim \mathbb{C}(X)$, is ample. By [Mum77, Prop. 3.4 b)], $W^\text{can}_\mathbb{Z}^{\rho^m} \cong \Omega^d_{X^\mathbb{C}}(\log D)$, where $D := (X^\text{tor} - X^\text{red})$, descends to an ample line bundle over $X^\text{min}$. Hence, although the automorphic line bundle $W_\mathbb{Z}^{\rho^m}$ over $X$ is ample, its canonical extension $W^\text{can}_\mathbb{Z}^{\rho^m}$ over $X^\text{tor}$ is not ample unless the canonical morphism $X^\text{tor} \to X^\text{min}$ is an isomorphism.

One might guess that the subcanonical extension $W^\text{sub}_\mathbb{Z}^{\rho^m}$ over $X^\text{tor}$ (defined by projective and smooth cone decompositions) is ample, but subcanonical extensions are not preserved by tensor powers, and what can indeed be shown to be ample are more complicated. By Tai’s work (see [AMRT75, Ch. IV, Sec. 2]) and the observation first made in [LS11 property (5) preceding (2.1)] (see also [LS13 Prop. 4.2(5)]), there exist an integer $N_0$ and a (possibly nonreduced) normal crossings divisor $D'$ with $D'_\text{red} = D$ such that $(W^\text{can}_\mathbb{Z}^{\rho^m})^{\otimes N}(-D')$ is ample for all $N \geq N_0$, and this turned out to be the most useful positivity property for our purpose. (As a consequence, although $W^\text{can}_\mathbb{Z}^{\rho^m}$ is generally not ample, it is still nef and big.)

**Definition 3.1.** We say $\nu \in X^+_M$ is positive parallel if $\dim \mathbb{C}(W_\nu) = 1$ and if the pullback of $\nu$ to each $\mathbb{Q}$-simple factor $G'$ of $G_\mathbb{C}$ is a positive rational multiple of $\rho^{d'}$, for the corresponding factor $M'$ of $M$.

**Example 3.2.** Here are some examples of low ranks:

<table>
<thead>
<tr>
<th>$G$</th>
<th>all positive parallel weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}_2$</td>
<td>$\nu = k$ for $k \in \mathbb{Z}_{\geq 1}$</td>
</tr>
<tr>
<td>$\text{Res}_{F/\mathbb{Q}} \text{SL}_2, F/\mathbb{Q}$ real quadratic</td>
<td>$\nu = k(1, 1)$ for $k \in \mathbb{Z}_{\geq 1}$</td>
</tr>
<tr>
<td>$\text{SL}_2 \times \text{SL}_2$</td>
<td>$\nu = (k_1, k_2)$ for $k_1, k_2 \in \mathbb{Z}_{\geq 1}$</td>
</tr>
<tr>
<td>$\text{Sp}_4$</td>
<td>$\nu = k(1, 1)$ for $k \in \mathbb{Z}_{\geq 1}$</td>
</tr>
</tbody>
</table>

**Remark 3.3.** See [Lan16c Sec. 3] for a fairly complete description of the smallest positive parallel weights in all cases.

3.2. Vanishing for coherent cohomology.

**Theorem 3.4** (see [LS12 Thm. 8.7 and 8.20], [LS13 Thm. 8.13 and 8.23], and [Lan16c Thm. 4.1]). Let $\nu \in X^+_M$.

1. If there exists a positive parallel weight $\nu_-$ in $X^+_M$ such that $\nu + \nu_- \text{ is cohomological}$, then $H^i(X^\text{tor}, W^\text{can}_\nu) = 0$ for $i < d - l(w(\nu + \nu_-))$.

2. If there exists a positive parallel weight $\nu_+$ in $X^+_M$ such that $\nu - \nu_+ \text{ is cohomological}$, then $H^i(X^\text{tor}, W^\text{sub}_\nu) = 0$ for $i > d - l(w(\nu - \nu_+))$.

3. If there exist positive parallel weights $\nu_+$ and $\nu_-$ in $X^+_M$ such that $\nu - \nu_+$ and $\nu + \nu_- \text{ are both cohomological}$, then the interior cohomology $H^i_{\text{int}}(X^\text{tor}, W^\text{can}_\nu) := \text{im}(H^i(X^\text{tor}, W^\text{sub}_\nu) \to H^i(X^\text{tor}, W^\text{can}_\nu))$ satisfies $H^i_{\text{int}}(X^\text{tor}, W^\text{can}_\nu) = 0$ for $i \not\in [d - l(w(\nu + \nu_-)), d - l(w(\nu - \nu_+))].$
Remark 3.5. Using the dual BGG decompositions (2.3) and (2.3) (which replace the usual Hodge decompositions), Theorem 3.4 can be viewed as a Kodaira-type vanishing theorem for the coherent cohomology of automorphic bundles, although (in noncompact cases) it did not follow from vanishing results readily available in the literature of algebraic geometry. (See [LS13, Sec. 1–3] and [Suh].)

3.3. Vanishing for de Rham cohomology.

Theorem 3.6 (see [LS12, Thm. 8.16], [LS13, Thm. 8.18], and [Lan16c, Thm. 4.10]). Suppose \( \mu \in X^+_\mathbb{C} \) that is sufficiently regular in the sense that, for each \( \alpha \in \Phi^+_{G_{\mathbb{C}}} \), which comes from some \( \mathbb{C} \)-simple factor of \( G_{\mathbb{C}} \), we have:

\[
\langle \mu, \alpha^\vee \rangle \geq \begin{cases} 
0, & \text{if the factor is compact (i.e., roots are all in } \Phi_M); \\
1, & \text{if the factor is not compact and not of types } B \text{ or } C; \\
2, & \text{if the factor is not compact but is of types } B \text{ or } C.
\end{cases}
\]

Then:

1. \( H^i_{dR}(X, V^\vee_\mu) = 0 \) for \( i < d \).
2. \( H^i_{dR,c}(X, V^\vee_\mu) = 0 \) for \( i > d \).
3. The interior cohomology \( H^i_{dR, \text{int}}(X, V^\vee_\mu) := \text{im}(H^i_{dR,c}(X, V^\vee_\mu) \to H^i_{dR}(X, V^\vee_\mu)) \) satisfies \( H^i_{dR, \text{int}}(X, V^\vee_\mu) = 0 \) for \( i \neq d \).

Remark 3.7. This follows from the vanishing for coherent cohomology (using the dual BGG decompositions (2.3) and (2.3)), and (in noncompact cases) reproofs many Hermitian cases of Li and Schwermer’s result (see [LS04, Cor. 5.6]). (See [Lan16c, Rem. 4.16] for a more complete documentation of what were known earlier.) It is new (and beyond methods involving either automorphic or Galois representations) when \( \Gamma \) is not congruent, although our understanding of the cohomology of automorphic bundles in the noncongruent cases is still very limited.

3.4. Some examples.

Example 3.8. This is an example for Theorem 3.4 which is about vanishing for the coherent cohomology, when \( G = \text{Res}_{F/\mathbb{Q}} \text{SL}_2 \) with \( F/\mathbb{Q} \) real quadratic (which is the case of Hilbert modular surfaces). In the following diagram, starting with each weight represented by a bullet \( \bullet \), we shift it by both positive and negative multiples of \( (1,1) \) (which is the smallest positive parallel weight, as we have seen in Example 3.2), and consider the first cohomological weights we encounter in both directions, represented by two circles \( \circ \). Then we record the Weyl lengths \( l(w) \) associated with such cohomological weights, and determine the bounds \( d - l(w) \) for vanishing degrees (with \( d = 2 \)). For example, if we start with the weight \( (2,-1) \), then the first cohomological weight we encounter after shifting by a positive integral multiple of \( (1,1) \) is \( (3,0) \), which is of Weyl length \( l(w) = 1 \), and hence we obtain the vanishing of the cohomology of \( W^\text{can}_{(2,-1)} \) over \( X^\text{tor} \) below degree \( d - l(w) = 1 \). On the other hand, the first cohomological weight we encounter after shifting by a negative integral multiple of \( (1,1) \) is \( (0,-3) \), which is of Weyl length \( l(w) = 0 \), and hence we obtain the vanishing of the cohomology of \( W^\text{sub}_{(2,-1)} \) over \( X^\text{tor} \) above degree \( d - l(w) = 2 \) (which is useless because the cohomology of a surface always
vanish above degree $d = 2$). The cases for the weights $(3, 3)$ and $(-1, 0)$ are similar.

By considering all weights, we obtain the following summarizing diagram (see [Lan16c, Exer. 4.18] for more details), in which an interval $[a, b]$ (or simply $[a]$) means $H^i(X_{\text{tor}}, W_{\text{can}}) = 0$ for $i < a$, $H^i(X_{\text{tor}}, W_{\text{sub}}) = 0$ for $i > b$, and $H^i_{\text{int}}(X_{\text{tor}}, W_{\text{can}}) = 0$ for $i \notin [a, b]$:

- : regular and concentrate in one degree
- : $[0, 2]$ (useless)

Example 3.9. This is an example for Theorem 3.4, which is about vanishing for the de Rham cohomology, again when $G = \text{Res}_{F/Q} \text{SL}_2$ with $F/Q$ real quadratic. In the following diagram, the bullets $•$ denote the weights of the dual BGG pieces of the sufficiently regular weight $\mu = (1, 1)$, and the upshot is that these weights remain cohomological (with the same associated Weyl length) when shifted by the
smallest positive parallel weight \((1,1)\) and the opposite \((-1,-1)\).

In the following diagram, we apply the Weyl group action and move all weights to the cohomological region of Weyl length zero. Thus, the point of being sufficiently regular is that the weight then remains (up to duality) in the dominant chamber after shifting by the whole Weyl orbit of the smallest positive parallel weight.

**Example 3.10.** This is an example for Theorem 3.4 when \(\text{Lie } G_\mathbb{R} \cong \mathfrak{su}_{2,1}\) (which is the case of Picard modular surfaces; see [Lan16c, Exer. 4.19] for more details). (In
the following diagram, we visualize \((k_1, k_2; k_3) \mod (1, 1; 1)\) by \((k_1 - k_3, k_2 - k_3)\).）

- \(\bullet\): regular and concentrate in one degree by our theorem
- \(\square\): irregular but still concentrate in one degree by our theorem
- \(\circ\): \([0, 2]\) (useless)

**Example 3.11.** This is an example for Theorem 3.4 when \(G = \text{Sp}_4\) (which is the case of Siegle threefolds; see [Lan16c, Exer. 4.17] for more details).

- \(\bullet\): sufficiently regular
- \(\blacksquare\): regular and concentrate in one degree by our theorem
- \(\square\): irregular but still concentrate in one degree by our theorem
- \(\star\): regular and concentrate in one degree by Li–Schwermer
- \(\Delta\): \([0, 2]\)
- \(\triangledown\): \([1, 3]\)
- \(\circ\): \([0, 3]\) (useless)

**Example 3.12.** This is an example for Theorem 3.4 when \(\text{Lie } G \cong \mathfrak{e}_7(-25)\). In this case, we embed \(\Phi_{G_C}\) in \(\mathbb{R}^7\) (with Killing form induced by the Euclidean inner
product) with positive simple roots in $\Phi^+_\mathbb{C}$ give by

$$
\begin{align*}
\alpha_1 &= (1, -1, 0, 0, 0, 0), \\
\alpha_2 &= (0, 1, -1, 0, 0, 0), \\
\alpha_3 &= (0, 0, 1, -1, 0, 0), \\
\alpha_4 &= (0, 0, 0, 1, -1, 0), \\
\alpha_5 &= (0, 0, 0, 0, 1, -1), \\
\alpha_6 &= (0, 0, 0, 0, 1, 1), \\
\alpha_7 &= (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \sqrt{2}),
\end{align*}
$$

with $\alpha_1 \notin \Phi_M$. Then we have (see [Lan16c Exer. 5.46, 5.47, and 5.48]):

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>cohomological?</th>
<th>$\mu(\nu)$ regular?</th>
<th>$H^i_\text{int} = 0$ for?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(10, 10, 9, 7, 4, 0, 26\sqrt{2})$</td>
<td>yes</td>
<td>yes</td>
<td>$i \neq 6$</td>
</tr>
<tr>
<td>$(-14, 8, 3, 2, 1, 0, \sqrt{2})$</td>
<td>yes</td>
<td>no</td>
<td>$i \notin [25, 27]$</td>
</tr>
<tr>
<td>$(-7, 5, 2, 1, 0, 3\sqrt{2})$</td>
<td>no</td>
<td>undefined</td>
<td>$i \notin [23, 24]$</td>
</tr>
</tbody>
</table>

3.5. **Relative vanishing.** Let $\pi : X^{\text{tor}} \to X^{\min}$ denote the canonical morphism. A rather unpredicted recent discovery about the cohomology of automorphic bundles is the following:

**Theorem 3.13** (see [LS14] and [Lan16c Thm. 4.5]). For every $\nu \in X^+_M$, we have $R^i\pi_*W^{\text{can}} = 0$ for all $i > 0$.

**Remark 3.14.** It is not true in general that $R^i\pi_*W^{\text{can}} = 0$ for all $i > 0$.

**Remark 3.15.** The first proofs of results like Theorem 3.13 (see, for example, [Hab16 Thm. 5.4], [Lana Sec. 8.2], and [Lan16b Thm. 3.9 and Rem. 10.1]) were based on detailed analyses of the formal fibers of $\pi$. The later (shorter) proofs in [LS14] and [Lan16c Thm. 4.5] were based on (various versions of) Theorem 3.4.

3.6. **Higher Koecher’s principle.** The relative vanishing of Theorem 3.13 implies the following generalization of the classical Koecher’s principle, whose discovery was surprisingly late, given how fundamental the assertion seems to be:

**Theorem 3.16** (higher Koecher’s principle; see [Lan16b Thm. 2.5] and [Lan16c Thm. 4.7]). Let $\nu \in X^+_M$. Let $c_X := \text{codim}_X(X^{\min} - X, X^{\min})$.

Let us denote by $j^{\text{tor}} : X \hookrightarrow X^{\text{tor}}$ and $j^{\min} : X \hookrightarrow X^{\min}$ the canonical morphisms. Then the canonical morphism (induced by $j^{\text{tor}}$)

$$R^i\pi_*W^{\text{can}} \to R^i j_*^{\text{can}}W,$$

is an isomorphism for $i < c_X - 1$, and is injective for $i = c_X - 1$. Hence, by the Leray spectral sequence, for any open $U$ in $X^{\min}$, the canonical restriction

$$H^i(\pi^{-1}(U), W^{\text{can}}) \to H^i((j^{\min})^{-1}(U), W),$$

is bijective (resp. injective) for all $i < c_X - 1$ (resp. $i = c_X - 1$).

**Remark 3.17.** This theorem implies its complex analytic analogue (by the same argument as in [Lan16b Sec. 3]).

**Remark 3.18.** When $i = 0$, $U = X^{\min}$, and $c_X > 1$, this specializes to the classical Koecher’s principle, which asserts that a section of $W$ over $(j^{\min})^{-1}(U) = X$ automatically extends to a section of $W^{\text{can}}$ over all of $\pi^{-1}(U) = X^{\text{tor}}$, or in other words that the growth condition is unnecessary in the definition of such (generalizations of) modular forms. (See [Lan16b Sec. 2].)
Example 3.19. Applying Theorem 3.16 with \( U = X_{\min} \), we see that the canonical restriction map
\[
H^i(X_{\tor}, W^\can_\nu) \to H^i(X, W_\nu)
\]
is bijective if \( i < c_X - 1 \), and injective if \( i = c_X - 1 \). For the values of \( c_X \), we have the following summarizing table:

<table>
<thead>
<tr>
<th>( G ) (most split form)</th>
<th>( \text{rk}_G )</th>
<th>( \text{dim}_C(X) )</th>
<th>( \text{dim}<em>C(X</em>{\min} - X) )</th>
<th>( c_X )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{SL}_2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \text{SL}_2, F/\mathbb{Q} \text{ totally real} )</td>
<td>1</td>
<td>( d = [F : \mathbb{Q}] )</td>
<td>0</td>
<td>( d )</td>
</tr>
<tr>
<td>( \text{Sp}_4 )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( \text{Sp}_{2n}, n \geq 1 )</td>
<td>( n )</td>
<td>( \frac{1}{2}n(n+1) )</td>
<td>( \frac{1}{2}n(n-1) )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{su}</em>{2,1} )</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{su}</em>{n,1}, n \geq 1 )</td>
<td>( n )</td>
<td>( a )</td>
<td>( (a-1)(b-1) )</td>
<td>( a+b-1 )</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{su}</em>{a,b}, ab \geq 1 )</td>
<td>( \text{min}(a,b) )</td>
<td>( ab )</td>
<td>( a+b-1 )</td>
<td>( a+b-1 )</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{so}</em>{1,2} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{so}</em>{n,2}, n \geq 2 )</td>
<td>( \frac{1}{2} )</td>
<td>( n )</td>
<td>( 1 )</td>
<td>( n-1 )</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{so}</em>{2n}, n \geq 2 )</td>
<td>( \frac{1}{2} )</td>
<td>( n(n-1) )</td>
<td>( \frac{1}{2}(n-2)(n-3) )</td>
<td>( 2n-3 )</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{e}</em>{(-14)} )</td>
<td>2</td>
<td>16</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>( \text{Lie}<em>G \cong \mathfrak{e}</em>{*(25)} )</td>
<td>3</td>
<td>27</td>
<td>10</td>
<td>17</td>
</tr>
</tbody>
</table>

Remark 3.21. If \( i = c_X - 1 \), then (3.20) can actually fail to be surjective, up to shifting \( \nu \) by a sufficiently large multiple of \( 2\rho^M \) (by the same argument as in \[Lan16b\] Sec. 9)). This suggests that there are some mysterious fake modular forms in degree \( c_X - 1 \). But we have no idea what they are—the proof of their existence is based on \[Gro68\] VIII, Prop. 3.2], a general fact in the theory of local cohomology.

3.7. Vanishing over integral models. To prove the above theorems with coefficients over integers or their reductions modulo integers, we can no longer just work with the complex analytic locally symmetric varieties—we need certain good integral models. (We are being vague here about what “good” means.)

So far all known constructions of good integral models rely on an important coincidence: When \( G = \text{Sp}_{2n} \) or better \( \text{GSp}_{2n} \) (to avoid introducing roots of unity into the base rings), \( X \) is a moduli space of principally polarized abelian varieties with some level structures over \( \mathbb{C} \). This led to the construction of good integral models of \( X \) for the PEL- and Hodge-type Shimura varieties, by taking normalizations over moduli of abelian schemes with PEL structures (i.e., polarizations, endomorphism structures, and level structures) and with certain Hodge tensors. (See, for example, \[Kot92\] and \[Lan13\] for the construction of smooth PEL moduli; see \[Kis10\] for the construction of good reduction integral models of Hodge-type Shimura varieties; and see \[Lan16a\] for the construction by normalization in all PEL-type cases, allowing bad reduction.) We now know how to also construct good integral models of \( X_{\tor} \) and \( X_{\min} \) in such cases, allowing arbitrary ramifications, levels, polarization degrees, and isogeny collections. (See \[Lan13\], \[Lan12c\], \[Lan12a\], \[Lan16a\], \[Lan15\], \[Lanc\], and \[Lanb\] for various constructions and comparisons in PEL-type cases; and see \[MP15\] for a different approach in Hodge-type cases.) The construction of good reduction integral models of \( X \) extends to the abelian-type cases—see \[Kis10\] again. In \[KP15\], the construction is further extended to allow bad reductions with parahoric level structures. (The construction for \( X_{\tor} \) and \( X_{\min} \) should also extend, but not carried out yet.)
At least in PEL-type cases, the automorphic bundles $W_{\nu}$ and their extensions $W_{\nu}^{\mathrm{can}}$ and $W_{\nu}^{\mathrm{sub}}$ can also be defined over even bad reduction integral models of $X$. In good reduction cases, we can also consider $(V_\mu, \nabla)$ over smooth integral models of $X$, their extensions $(V_{\mu}^{\mathrm{can}}, \nabla)$ and $(V_{\mu}^{\mathrm{sub}}, \nabla)$ over smooth integral models $X^\mathrm{tor}$ with log poles over $D = (X^\mathrm{tor} - X)_{\text{red}}$, and their (log) de Rham cohomology; and we have $p$-torsion or $p$-integral analogues of Theorems 3.4 and 3.6 when $p$ is unramified and larger than a bound independent of the level away from $p$. The short exact sequence $0 \to W_{\nu}^{\mathrm{can}} \to W_{\nu}^{\mathrm{can}} \to W_{\nu}^{\mathrm{can}} \to 0$ induces long exact sequences $\cdots \to H^i(W_{\nu}^{\mathrm{can}}) \to H^i(W_{\nu}^{\mathrm{can}}) \to H^i(W_{\nu}^{\mathrm{can}}) \to \cdots$. Hence, vanishing mod $p$ induce not just vanishing over $\mathbb{Z}_p$, but also freeness and liftability. By crystalline comparison (see [LS12, Sec. 5] and [LS13, Sec. 9], and the references there), under certain assumptions on $p$ (and on the local properties of the integral models), the vanishing of (log) de Rham cohomology mod $p$ implies the vanishing of the corresponding étale and Betti cohomology with coefficients mod $p$ (which then also implies the corresponding freeness and liftability assertions). (For more details, see [LS12], [LS13], and [LP]; see also the survey article [Lan12b].)

On the other hand, the relative vanishing in Theorem 3.13 has analogues over integral models of PEL-type Shimura varieties even in bad reduction cases (see, for example, [Lanc, Thm. 8.6] and [Lanb, Thm. 4.4.9]), which are important because, roughly speaking, many new techniques for congruences involve the consideration of the sheaves $\pi_{\nu} W_{\nu}^{\mathrm{sub}}$ over $X^\mathrm{min}$. However, in such generality, we have to resort to some detailed analyses of formal fibers of $\pi$, because we no longer expect any Kodaira-type vanishing as in Theorem 3.4 to be true (cf. Remark 3.15). Thus, the nature of the relative vanishing as in Theorem 3.13 remains mysterious.

4. Application to the construction of Galois representations

4.1. Conjectural framework and historical developments. Let us begin with the following conjectural framework (incorporating conjectures due to Langlands, Clozel [Clo90], Fontaine–Mazur [FM97], and some others). Given any prime number $p$, any field isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$, any number field $F$, and any $n \in \mathbb{Z}_{\geq 1}$, it is conjectured that there is a natural bijection:

\[
\begin{cases}
\text{irreducible algebraic cuspidal} \\ \text{continuous representations} \\ \pi \cong \otimes_v \pi_v \text{ of } \GL_n(\mathbb{A}_F) \\
\end{cases}
\leftrightarrow
\begin{cases}
\text{irreducible algebraic} \\
\text{r : } \Gal(\overline{F}/F) \to \GL_n(\mathbb{Q}_p) \\
\end{cases}
\]

The term algebraic at the left-hand side means the Harish-Chandra parameter of the Archimedean component $\pi_\infty = \otimes_v |\cdot|^s \pi_v$ is integral after shifting by the half-sum of positive roots. The term algebraic at the right-hand side means $r$ is unramified at all but finitely places and de Rham at $p$. (These are often called geometric because they are satisfied by the $p$-adic étale cohomology of varieties over $\mathbb{Q}$.) The bijection is natural in the sense that it matches the local $L$-factors on both sides—or it suffices to know the local-global compatibility at all finite places $v \nmid p$ over which $F$, $\pi_v$, and $r_v$ are all unramified, by matching (under $\iota$) the Satake parameters of $\pi_v$ with the Frobenius eigenvalues of $r_v$ (up to some normalization).

Here is a summary of some historical developments:

- When $n = 1$, the conjecture is known by class field theory.
When \( n > 1 \), the conjecture is often studied in two directions: attaching Galois representations \( r \) to automorphic representations \( \pi \), and the converse (i.e., modularity or automorphy problems).

When \( F \) is CM or totally real, for cohomological (i.e., regular algebraic) and polarized \( \pi \), one can construct \( r \) by traditional methods: by realizing \( r \) (up to base change and patching) in the étale cohomology of Shimura varieties when \( \pi_\infty \) has sufficient regular weights (which is often proved by trace formula techniques), and by extending such \( r \) to other cohomological weights by congruences. (One can further extend the construction to cases where \( \pi_\infty \) is some holomorphic limits of discrete series, realized in \( H^0 \), by congruences; see, for example, [Tay91].)

The modularity or automorphy problems are often studied by the so-called Taylor–Wiles method. For the method to work in higher dimensions, one often needs the cohomology to behave like a free module over some auxiliary Hecke algebras, and this requires rather strong vanishing results.

### 4.2. Removal of polarization condition.

**Theorem 4.1** (Harris–Lan–Taylor–Thorne; see [HLTT16, Introduction, Thm. A]).

Let \( p, \iota: \mathbb{Q}_p \to \mathbb{C} \), and \( n \) be as above, and \( F \) totally real or CM. Suppose \( \pi \) is a cohomological cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}_F) \) (which is then necessarily algebraic). Then there is a unique semisimple representation \( r = r_{p,\iota}(\pi): \text{Gal}(\overline{F}/F) \to \text{GL}_n(\mathbb{Q}_p) \) such that, if \( q \neq p \) is a prime above which \( \pi \) is unramified and if \( v|q \) is a place of \( F \), then \( r \) is unramified at \( v \) and satisfies the local-global compatibility

\[
r|_{W_{F_v}}^{ss} = \iota^{-1} \text{rec}_{F_v}(\pi_v | \det |_v^{\frac{1}{n}}).
\]

**Remark 4.2.** The most important point of Theorem 4.1 is that we impose no polarization condition on \( \pi \) (i.e., not requiring \( \pi \) to be conjugate self-dual up to character twists). Then \( r \) generally does not occur in the \( p \)-adic étale cohomology of any Shimura variety!

Instead, we constructed \( r \) using \( p \)-adic limits of Galois representations which do occur in the \( p \)-adic étale cohomology of some Shimura varieties. The main ideas can be briefly (and somewhat imprecisely) summarized as follows:

1. We may assume that \( n > 1 \). Also, by patching, we may assume that \( F \) is CM and satisfies certain simplifying assumptions.
2. The starting point is Skinner’s idea that, to construct \( r = r_{p,\iota}(\pi) \), it suffices to construct \( R_t = r_{p,\iota}(\pi | \det |_t^{\frac{1}{n}}) \oplus r_{p,\iota}(\pi | \det |_t^{\frac{1}{n}})^{c \vee} \) for sufficiently many \( t \).
3. Each \( R_t \) should correspond to Eisenstein series for “GU(\( n, n \))” (with Levi \( \text{GL}_n, F \times \mathbb{G}_m \)). But nothing along that line really works.
4. Rather, we construct them as overconvergent cusp forms, which are global sections of \( \pi_* W_{\text{sub}}^{\dagger} \) (i.e., the overconvergent version of \( \pi_* W_{\text{sub}} \)) over the affinoid (multiplicative-type) ordinary locus \( X^{\text{ord, min}, \dagger} \) of \( X^{\text{min}, \dagger} \) for some \( \nu \in X^+_M \), for some “GU(\( n, n \))” Shimura varieties \( X \). (Here the superscripts \( \text{ord} \) and \( \dagger \) mean the dagger spaces attached to the tubes defined by certain \( p \)-integral models with only ordinary loci in characteristic \( p \).) They are naturally \( p \)-adic limits of cusp forms as coherent cohomology classes whose associated Galois representations were already known (thanks to historical
developments in the polarized case, because we are working on Shimura varieties associated with unitary similitude groups over CM fields).

(5) Using the mysterious relative vanishing $R^i\pi_*W^\text{sub,}^\dagger = 0$ for $i > 0$, it suffices to construct sections of $W^\text{sub,}^\dagger$ over the (non-affinoid) ordinary locus $X^\text{ord,tor,}^\dagger$ of $X^\text{tor,}^\dagger$, and we achieved this using the Hodge spectral sequence of certain rigid cohomology of $X^\text{ord}$ with compact support along the partial boundary $X^\text{ord,tor} - X^\text{ord}$, or of the analogue with $X$ and $X^\text{tor}$ replaced with certain Kuga families (which are self-fiber-products of the universal abelian schemes over the original $X$) and their toroidal compactifications. (Here the overlines and the superscripts $\text{ord}$ mean the characteristic $p$ fibers of certain integral models with only ordinary loci, the same ones used in the definition of dagger spaces above.)

(6) The key observation is the following: The partial boundary $X^\text{ord,tor} - X^\text{ord}$ and its analogue for Kuga families can be arranged to be simple normal crossings divisors such that the incidence relations of their smooth irreducible components are essentially the same as in characteristic zero, which encode the interior cohomology of the (real analytic) locally symmetric manifold associated with $\GL_n,F$ (with coefficients in local systems associated with finite-dimensional representations of $\GL_n,F$ of polynomial weights). Then a Frobenius-weight argument (based on [Ber97, Thm. 3.1] and [Chi98, Thm. 2.2]) shows that such interior cohomology contributes to the rigid cohomology mentioned above. (Note that we are really realizing the interior cohomology of a real analytic manifold in the rigid cohomology of some characteristic $p$ algebraic variety.) But all $\pi$ considered in Theorem 4.1 (or more precisely their non-Archimedean components) contribute up to determinant twists (and $\iota$) to such interior cohomology!

Remark 4.3. In [Sch15], Scholze reproved Theorem 4.1 using a much more advanced method, and also treated the analogue for torsion cohomological classes—in fact, the consideration of torsion classes is crucial in his argument. Nevertheless, to show that the Galois representation $\rho$ constructed in Theorem 4.1 is indeed algebraic, at least with current techniques, it still seems easier to use the results in [HLTT16].

There have been many other exciting developments since [HLTT16] and [Sch15], but it is not easy to give a fair overview of all of them, and we decided to stop here due to limitation of time and energy.

References


[Har90] , Automorphic forms and the cohomology of vector bundles on Shimura varieties, in Clozel and Milne [CM90b], pp. 41–91.


