Practice Problems 1

Math 5467

1. Basic Concepts of Digital imaging

Solve problems 2.9 and 2.10 of the textbook. In 2.10: "resolution of 1125 horizontal TV lines interlaced..." means "there are 1125 pixels in the vertical direction".

2. Linear Algebra Review

You may solve only 2 out the following 3 questions.

a) Prove that if A is an $n \times n$ (real) symmetric matrix, then there exists an $n \times n$ (real) orthogonal matrix U and $n \times n$ (real) diagonal matrix D, such that $A = U \cdot D \cdot U^T$.

b) Given

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} ,$$

plot the regions

$$\{Ax \mid x \in \mathbb{R}^2 \text{ and } \|x\|_2 = 1\} ,$$

and

$$\{Ax \mid x \in \mathbb{R}^2 \text{ and } \|x\|_2 \leq 1\} .$$

Explain your solution.

c) Let A be an $m \times n$ matrix with elements $\{a_{i,j}\}_{1 \leq i \leq m, 1 \leq j \leq n}$; express the quantity $\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}^2$ as a function of the eigenvalues of the matrix $A \cdot A^T$.

3. Basic Matlab for Images

Report your work by printing your Matlab commands and output.

a) Load a color image (any of your favorite which is appropriate to hand in...) having more than 512 pixels in both rows and columns. Create a corresponding 512×512 gray-level image out of it (there are many ways of doing it).

b) Draw a 3-D graph of the function corresponding to the image (use the commands *meshgrid* and *meshc*).

c) By manipulating the associated matrix create an up-side-down as well as left-to-right corresponding images.

d) Subsample the image by a factor of 2, five times, i.e., by factors of 2,4,...,32, and show the resulted images in one figure (use the subplot command).

4. Computing the Singular Value Decomposition and the Pseudo-inverse

a) Let A be the 1000×2 matrix satisfying the following conditions:

$$(A)_{i,j} = \begin{cases} 1 & \text{if } j = 1, \\ 2 & \text{if } j = 2 \text{ and } i \text{ is odd}, \\ 0 & \text{if } j = 2 \text{ and } i \text{ is even}. \end{cases} \tag{1}$$

Compute directly the thin SVD of A (do not use Matlab, but you may use calculator to approximate the irrational numbers; also no need to specify all numbers of the matrix U, but just provide a formula for computing them).

b) Use Matlab to compute the thin SVD and print the matrices S and V only. If they are different than yours, then explain whether the differences are acceptable.

c) Describe the following set in $\mathbb{R}^{1000}$:

$$\{Ax \mid x \in \mathbb{R}^2, \quad \|x\|_2 = 1\}.$$

d) Using the appropriate formulas and your answer above (not Matlab), describe the pseudo-inverse of the 1000×2 matrix defined in equation (1); your description could be formulated in a similar way to that of A in equation (1).

e) Use Matlab to verify your answer of part d) and print your commands and short output (do not print the pseudo-inverse). Guide: compare the norms of Matlab's output and your output; you may use the command *pinv* to compute the pseudo-inverse.

5. Least Squares Solutions of Linear Systems

a) If A is the matrix defined in equation (1) and b and c are $1000 * 1$ vectors whose elements are

$$b_i = \begin{cases} 2 & \text{if } i \text{ is odd}, \\ 4 & \text{if } i \text{ is even}, \end{cases}$$

and

$$c_i = \begin{cases} 1 & \text{if } i = 1 \bmod 4, \\ 2 & \text{if } i = 2 \bmod 4, \\ 3 & \text{if } i = 3 \bmod 4, \\ 4 & \text{if } i = 0 \bmod 4. \end{cases}$$

Find the least squares solutions of the systems $A \cdot x = b$ and $A \cdot x = c$ (do not use Matlab, but solve by hand).

b) If x is the least squares solution of $A \cdot x = b$, find the l_2 distance of $A \cdot x$ from b. Similarly, if x is the least squares solution of $A \cdot x = c$, find the l_2 distance of $A \cdot x$ from c (do not use Matlab, but solve by hand).

c) Use Matlab to verify your answers for both parts and print the output (you may use the command *regress*).

6. Bonus Problem: On a Property of Singular Values
This is a bonus problem, you do not need to submit it.

We denote by $\sigma_i(B)$ the i-th singular value of B (sorted in descending order). Prove that if A_1 and A_2 are $m \times n$ matrices, then for all i and j in $\mathbb{N}$:

$$\sigma_{i+j-1}(A_1 + A_2) \leq \sigma_i(A_1) + \sigma_j(A_2).$$