

Practice Problems 4

Math 5467

1. DFT in Practice

You will need to download the image *granular_media.jpg*, which is supplemented next to the homework. The answer to subquestions a)-c) should be in the form of printed Matlab commands and figures.

a) Compute the DFT of the 200×200 subimages of *granular_media.jpg* created as follows

```
X = imread('granular_media.jpg');  
X1 = double(X(501:700,1401:1600));  
X2 = double(X(1:200,1:200));
```

Shift the Fourier transform so that it is centered around the middle of your figure and plot the absolute value of the shifted DFT (the spectrum). Then apply the log transform ($\log(1 + |DFT|)$) and plot your result. Remarks: use the command *fft2* and *fftshift*; If you use the command *imshow* make sure to write *imshow('image_name',[])*, alternatively you may use the command *imagesc*.

b) Recognize the most significant δ -signals in both DFTs (by observing the spectrum) and filter those δ -signals, so the rest of pixels get zero values. For example, you may use the following commands for the subimage *X1* (note that the parameter *f* is arbitrary, but make sure to use the same value for both *X1* and *X2*):

```
F1 = fft2(X1);  
Fc1 = fftshift(F1);  
f = 3/4;  
large_idx_Fc1 = log(abs(Fc1)+1) > f * max(max(log(abs(Fc1)+1)));  
Fc_deltas = Fc1.*large_idx_Fc1;
```

c) Compute the inverse DFT of the images obtained in part b) (make sure to invert the original shift by applying *ifftshift* first and then *ifft2*).

d) Assume a two-dimensional signal $x(n_1, n_2)$, $0 \leq n_1 \leq N - 1$, $0 \leq n_2 \leq M - 1$, which is a mixture of two δ -signals. One at the point (k_1, k_2) , where $1 \leq k_1 \leq N - 1$ and $1 \leq k_2 \leq M - 1$ and another at the point $(N - k_1, M - k_2)$. Compute the inverse DFT of that signal.

e) Use your computation from part d) to explain your results from part c) (note that we did not shift the image in part d).

2. More Properties of the FT and DFT

a) Show that if R is a 2×2 orthogonal transformation then the 2-dimensional Fourier transform of the function f (assume e.g. $f \in L_1(\mathbb{R}^2)$) is invariant under the transformation R . That is, if $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$, $\underline{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$, and $\hat{f}(\underline{\xi})$ denotes the Fourier transform of $f(\underline{x})$, then the Fourier transform of $f(R \cdot \underline{x})$ is $\hat{f}(R \cdot \underline{\xi})$.

b) Show that if $N \in \mathbb{N}$, $x, y \in \mathbb{C}^N$, and \hat{x}, \hat{y} are the DFT of x and y respectively, then

$$\langle x, y \rangle = \frac{1}{N} \langle \hat{x}, \hat{y} \rangle,$$

and

$$\|x\|_2 = \frac{1}{\sqrt{N}} \|\hat{x}\|_2.$$

3. The Fast Fourier Transform

Show that if $\{x(m)\}_{m=1}^N$ is an N -periodic signal and $N = 3^n$, then the DFT of $\{x(m)\}_{m=1}^N$ can be computed by no more than $5N \log_3 N$ operations.

4. Questions from the textbook

Solve problems 4.18 (the answer should be $N \cdot \delta_0$ instead of δ_0), 4.19, 4.21, 4.22 (you may use the result you proved in problem 2b), 4.23, 4.25a, b, 4.27a, 4.29, 4.33, 4.34.

5. Bonus Problem: f and \hat{f} Cannot Have Compact Support Simultaneously

Show that there is no function f in $L_1(\mathbb{R})$ such that both f and \hat{f} have compact support (a function f has a compact support if there exists a positive number a such that for any $x \in \mathbb{R}$ with $|x| > a$: $f(x) = 0$, that is, the function is zero outside a compact interval).

6. Bonus Problem: Padding

Show that in order to avoid wraparound error in 2-D convolution of two images with sizes $A \times B$ and $C \times D$, one can pad those images with zeros so that their mutual sizes become $P \times Q$, where $P \geq A + C - 1$ and $Q \geq B + D - 1$.

7. Bonus Problem: FFT

Assume an N periodic signal, where $N = k^n$, $k, n \in \mathbb{N}$. Establish an efficient upper bound on the number of operations to compute the DFT of such a signal.