Practice Problems 5  
Math 5467 (Spring 2013)  
Due Tuesday April 30th

1. Elementary Properties of Haar Wavelets

Throughout this exercise $\phi$ is the Haar scaling function and $\psi$ is the Haar wavelet function, i.e., $\phi = \chi_{[0,1)}$ and $\psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$.

You may solve only 5 out of the following 6 subquestions.

Remark: Some of the following properties have been proved (or will be proved) for general wavelets and MRAs, but here you need to prove them directly for the Haar wavelets and MRA.

a) Show that
$$\phi_{j,k}(x) = \frac{1}{\sqrt{2}} (\phi_{j+1,2k}(x) + \phi_{j+1,2k+1}(x))$$
and
$$\psi_{j,k}(x) = \frac{1}{\sqrt{2}} (\phi_{j+1,2k}(x) - \phi_{j+1,2k+1}(x)).$$

b) If $c_0 = (c_0(0), \ldots, c_0(N-1))$ is a signal of length $N = 2^n$, we associate with it the function
$$f(x) = \sum_{k=0}^{N-1} c_0(k) \phi_{n,k}(x),$$
so that $c_0(k) = \langle f, \phi_{n,k} \rangle$, for $k = 0, \ldots, N - 1$. We define for $j = 1, \ldots, n$
$$c_j(k) = \langle f, \phi_{n-j,k} \rangle$$
and
$$d_j(k) = \langle f, \psi_{n-j,k} \rangle.$$

Show that for $j = 1, \ldots, n$:

$$
\begin{pmatrix}
  c_j(k) \\
  d_j(k)
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
  1 & 1 \\
  1 & -1
\end{pmatrix} \begin{pmatrix}
  c_{j-1}(2k) \\
  c_{j-1}(2k+1)
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
  c_{j-1}(2k) \\
  c_{j-1}(2k+1)
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
  1 & 1 \\
  1 & -1
\end{pmatrix} \begin{pmatrix}
  c_j(k) \\
  d_j(k)
\end{pmatrix}.
$$

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\( c) \ We \ assume \ an \ arbitrary \ signal \ of \ length \ N = 4 \ (n = 2) \ with \ given \ coefficients \ vector \ \left( c_0(0), c_0(1), c_0(2), c_0(3) \right). \\
Write \ down \ a \ matrix \ which \ when \ multiplying \ this \ vector \ gives \ \left( c_1(0), c_1(1), d_1(0), d_1(1) \right) \ and \ another \ matrix \ that \ when \ multiplying \ the \ same \ original \ vector \ gives \ the \ Haar \ DWT \ \left( c_2(0), d_2(0), d_1(0), d_1(1) \right). \\
d) \ We \ assume \ an \ arbitrary \ signal \ of \ length \ 8 \ as \ above \ (n = 3) \ with \ the \ coefficients \ vector \ \left( c_0(0), c_0(1), \ldots, c_0(7) \right). \\
Write \ down \ an \ 8 \times 8 \ matrix \ which \ when \ multiplying \ this \ vector \ gives \ \left( c_1(0), \ldots, c_1(3), d_1(0), \ldots, d_1(3) \right) \ and \ another \ 8 \times 8 \ matrix \ that \ when \ multiplying \ the \ same \ original \ vector \ gives \ the \ Haar \ DWT \ \left( c_3(0), d_3(0), d_2(0), d_2(1), d_1(0), \ldots, d_1(3) \right). \\
\( e) \ Assume \ a \ general \ signal \ of \ length \ N = 2^n, \ where \ n \in \mathbb{N}. \ Show \ that \ the \ kth \ row \ (k = 0, \ldots, N-1) \ of \ the \ N \times N \ matrix \ representing \ the \ Haar \ DWT \ is \ given \ by \ h_k(z) \ for \ z = 0/N, \ 1/N, \ 2/N, \ \ldots, \ (N-1)/N, \ where \ h_0(z) = 1/\sqrt{N} \ for \ all \ z \in [0,1] \ and \ for \ k = 1, \ldots, N-1 \ we \ write \ k = 2^p + q - 1, \ where \ 0 \leq p \leq n - 1, \ 1 \leq q \leq 2^p, \ and \ for \ z \in [0,1]: \\
h_k(z) = h_{pq}(z) = \begin{cases} 
2^p/2, & \text{if } (q-1)/2^p \leq z < (q-0.5)/2^p; \\
-2^p/2, & \text{if } (q-0.5)/2^p \leq z < q/2^p; \\
0, & \text{otherwise.}
\end{cases} \\
f) \ Show \ that \ the \ computation \ of \ the \ Haar \ DWT \ of \ a \ signal \ of \ length \ N \ takes \ at \ most \ 4 \cdot N \ operations. \\

2. Another example of a wavelet \\
You \ may \ solve \ only \ 3 \ out \ of \ the \ following \ 4 \ subquestions. \\
Remark: \ It \ will \ be \ useful \ to \ verify \ the \ following \ properties \ in \ the \ Fourier \ domain. \\
a) \ Let \ \phi(x) = \text{sinc}(x) = \sin(\pi x)/(\pi x). \ Prove \ that \ for \ each \ j \in \mathbb{Z} \ the \ space \ V_j \ \text{(corresponding \ to \ the \ scaling \ function \ \phi)} \ \text{is \ the \ space \ of \ all \ band \ limited \ functions \ in} \ L_2 \ \text{with \ band} \ [-2^{j-1}, 2^{j-1}]. \ \text{That \ is, \ the \ space \ of \ functions \ whose \ fourier \ transforms \ \text{are \ in} \ L_2 \ \text{and \ supported \ within \ the \ interval} \ [-2^{j-1}, 2^{j-1}]. \\
b) \ Show \ that \ \phi \ is \ a \ good \ scaling \ function. \ That \ is, \ it \ gives \ rise \ to \ an \ MRA, \ which \ is \ called \ Shannon’s \ MRA. \\
c) \ Let \ \psi \ be \ the \ function \ such \ that \\
\hat{\psi}(\xi) = -e^{-\pi i \xi} \chi_I, \ \text{where} \ I = \left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \\
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(one can show that $\psi(x) = \text{sinc}(x)(1 - 2\sin(\pi x))$, but this is not part of this exercise). Show that

$$V_1 = V_0 \bigoplus W_0.$$  

Note that you need to show both that $V_1 = V_0 + W_0$ and that the space $W_0$ is orthogonal to the space $V_0$.

d) Write the MRA and the wavelet equations for the Shannon wavelet and scaling function. That is, find the coefficients $h_\phi$ and $h_\psi$.

Hint: Use Shannon’s sampling Theorem. Note that the corresponding filters (represented by $h_\phi$ and $h_\psi$) have infinite response.

3. Problems from the textbook

Solve problems 7.1, 7.2, 7.9, 7.11, 7.12, 7.13, 7.14, 7.15

4. Block-matching implementation of the non-local means (by Bryan Poling)

The first step of the denoising algorithm will be to build a “patch dictionary”. This will contain an object for each pixel in the image to represent a small image patch centered at that pixel. We will use patches of size $n$ pixels by $n$ pixels (we will only use odd integers for $n$). A patch object will be a vector of size $3n^2$. The elements of this vector will be the Red, Green, and Blue components of each pixel in the patch. The ordering is not important, so long as you pick an order and stick with it. Note that you may apply a code provided in class earlier [http://www.math.umn.edu/~lerman/math5467/matlab/image_patches_conversion.zip](http://www.math.umn.edu/~lerman/math5467/matlab/image_patches_conversion.zip), but make sure to carefully comment (with explanation) the part of the code you use.

After you have the patch dictionary, it is time to run through the main denoising algorithm. Iterate over each pixel in the image, $c$, and grab the vector from your patch dictionary that corresponds to $c$. We will call this vector $v_c$. For each pixel, $p$, within a radius of 15 pixels of $c$, compute the Euclidean distance between $v_c$ and $v_p$, where $v_p$ is the vector corresponding to pixel $p$ in your patch dictionary. The distance $\|v_c - v_p\|$ will be called the “patch distance” between pixels $c$ and $p$. Let $\{p_i\}$ be the set of 3 pixels that had the smallest patch distances from $p$ (including $p$ itself). Let $c'$ equal the average value of these 3 pixels, component-wise (this means average each color channel separately). Replace each pixel intensity $c$ with $c'$. After iterating over all pixels in your image, save your denoised image to a file.

Submit your Matlab source code for this homework (the patch dictionary code given in class also uses a C code to speed it up and it is okay to do this, if you also provide the C code). In addition, run your algorithm on the provided image with $n = 3$ and include the results, presented next to the original, noisy image. You should try to make your code readable. This means that you should select descriptive variable names, like patchDictionary, n, etc. Also, block matching is expensive and Matlab is not designed for speed. Don’t be surprised if your program takes a long
time to run. There is also a question about how to handle pixels along edges of the image, where you cannot get proper image patches. You must decide on a reasonable way to handle these pixels.

5. Bonus Problem: Another Sampling Theorem

Prove that if $M \in \mathbb{N}, N = 2M + 1$ and

$$f(t) = \sum_{|n| \leq M} c_n e^{2\pi i n t},$$

then

$$f(t) = \frac{\sin(\pi t N)}{N} \sum_{m=0}^{N-1} f\left(\frac{m}{N}\right) \frac{(-1)^m}{\sin\left(\pi \left( t - \frac{m}{N} \right) \right)}.$$

6. Bonus Problem: Revisiting the MRA definition

Show that the third property defining MRA, i.e., $V_{-\infty} = \{0\}$, follows from the rest of the properties.