Things You Need to Know for the Final Exam

Math 5467  (Spring 2013)

The exam is cumulative and covers all material taught till and including the class of April 30th. Below are topics you need to know.

1. All questions given in homework assignments (5 worksheets).
2. Everything you needed to know for exams 1 and 2 (see previous 2 files).
3. What are image pyramids? Draw their diagrams.
4. a) Define an MRA for $L^2(\mathbb{R})$.
   Remark: Following the textbook start with a scaling function $\phi$, define the spaces $V_j$, $j \in \mathbb{Z}$, and then specify the MRA assumptions.
   b) Specify the Haar scaling function $\phi$, the corresponding functions $\phi_{j,k}(x)$ (know how to plot them for any $j$ and $k$) and the corresponding spaces \{\(V_j\)\}_{j \in \mathbb{Z}}. Describe in words the Haar spaces $V_j$, $j \in \mathbb{Z}$ (make sure to characterize them precisely).
   c) What is a scaling equation (equivalently an MRA equation)? Explain why it exists in an MRA.
   d) Derive the coefficients of the Haar MRA equation.
5. a) What do we mean by a wavelet function corresponding to an MRA? (make sure to define the spaces $W_j$, where $j \in \mathbb{Z}$, and indicate the relation between the spaces \{\(V_j\)\}_{j \in \mathbb{Z}} and \{\(W_j\)\}_{j \in \mathbb{Z}}.
   b) Specify the Haar wavelet function $\psi$, the corresponding functions $\psi_{j,k}(x)$ (know how to plot them for any $j$ and $k$) and the corresponding spaces \{\(W_j\)\}_{j \in \mathbb{Z}}.
   c) What is a wavelet equation? Explain why it exists.
   d) Derive the coefficients of the Haar wavelet equation.
6. a) What are the collection of dyadic intervals? and dyadic intervals of scale $2^{-j}$?
   b) Show directly (using properties of dyadic intervals) that if $\psi_I$ is the Haar wavelet associated with the dyadic interval $I$, then \{\(\psi_I\)\}_{I \in \mathcal{D}} is an orthonormal system in $L^2(\mathbb{R})$, where $\mathcal{D}$ denotes the
collection of all dyadic intervals.

c) Let $\phi_I$ denotes the Haar scaling function associated with the dyadic interval $I$. Show that

$$\{\psi_I\}_{I \in \bigcup_{j=j_0}^{\infty} D_j} \cup \{\phi_I\}_{I \in D_{j_0}}$$

is an orthonormal system in $L_2(\mathbb{R})$.

7. Assume an MRA with a scaling function $\phi$, wavelet function $\psi$, and corresponding spaces $\{V_j\}_{j \in \mathbb{Z}}$ and $\{W_j\}_{j \in \mathbb{Z}}$.

a) How do we express the space $L_2(\mathbb{R})$ by the spaces $W_j$, $j \in \mathbb{Z}$, and similarly any $f \in L_2(\mathbb{R})$ as an infinite linear combination of $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$? (specify a formula for the coefficients)?

b) How do we express $f \in L_2(\mathbb{R})$ by $\{\phi_{j_0,k}\}_{k \in \mathbb{Z}}$ and $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$? (specify the coefficients). Also, how do we express the function space $L_2(\mathbb{R})$ by MRA and wavelet subspaces in the same way of the function representation you have just described.

c) How do we express the MRA subspace $V_l$, for some $l \in \mathbb{Z}$, by the space $V_{j_0}$, for some $j_0 \in \mathbb{Z}$, where $j_0 < l$, and elements of $\{W_j\}_{j \in \mathbb{Z}}$? Also, write the representation of $f \in V_l$ by shifts and scales of $\phi$ and $\psi$ in the same way as the subspace representation you have just specified.

8. Using the scaling equation and the wavelet equation prove that

$$\phi_{j,k}(x) = \sum_{m \in \mathbb{Z}} h(m - 2k) \phi_{j+1,m} ,$$

and

$$\psi_{j,k}(x) = \sum_{m \in \mathbb{Z}} g(m - 2k) \phi_{j+1,m} .$$

9. a) Define the discrete wavelet transform (as done in class and not as in the textbook). Make sure to distinguish between the DWT of stage $j$ and the total DWT.

b) Prove that the DWT can be written in the form

$$c_{j+1}(k) = c_j * f_0(2k),$$

$$d_{j+1}(k) = c_j * f_1(2k).$$

Express $f_0$ and $f_1$ in terms of the scaling and wavelet filters $h$ and $g$. Also, draw a diagram corresponding to those formulas.

c) Write down the formula for the discrete Haar transform (expressing $c_{j+1}(k)$ and $d_{j+1}(k)$ in terms of coefficients of the form $c_j(k')$). Explain how it is obtained as a special case of the above equation.
13. Assuming that the number of coefficients for \( \{ h \} \) is \( K \) and similarly the number of coefficients for \( \{ g \} \) is \( K \). What is the number of operations for computing the first stage of the discrete wavelet transform of a signal of length \( M = 2^N \)? What is the number of coefficients of the total discrete wavelet transform of a signal of length \( M = 2^N \)? Prove your claim.