Research statements of Naichung Conan Leung

1. Recent and future work: Geometry of Calabi-Yau, Hyperkähler and G_2-manifolds

Much of my recent work [1, 2, 3, 4, 5, 6, 7, 8] and my current research is focused on understanding the geometry of Calabi-Yau manifolds, hyperkähler manifolds and G_2-manifolds, with emphasis on the mirror symmetry and other duality transformations.

In [1], a theory of Riemannian geometry over a normed division algebra is developed. It gives a *unified* approach to the studies of all possible holonomy groups. Furthermore, it also give a *unified* treatment of all different types of Yang-Mills bundles and calibrated submanifolds in these spaces. For example, Calabi-Yau, hyperkähler and G_2-manifolds correspond to special $\mathbb{A}$-manifolds with $\mathbb{A} = \mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively.

For Calabi-Yau manifolds, the Strominger–Yau-Zaslow conjecture says that the mirror transformation is a fiberwise Fourier transform on special Lagrangian fibrations on mirror manifolds. In [5], Vafa and I studied this construction in the toric case and generalized it to the local mirror symmetry.

In the semi-flat case, Yau, Zaslow and I showed in [4] that such a Fourier-Mukai transformation does take a special Lagrangian section to a deformed Hermitian-Yang-Mills connections on its mirror. To fully understand the mirror transformation, we also need the Legendre transformation along the base direction of the special Lagrangian fibration, as explained in [3] and [9]. In these papers, I combined the *Legendre transformation* and the *Fourier transformation* to demonstrate the mirror duality between the complex geometry and the symplectic geometry of semi-flat Calabi-Yau manifolds. Currently I am working on extending these transformations to general Calabi-Yau manifolds.

Hyperkähler manifolds constitute a special but important class of Calabi-Yau manifolds. In [2], I established the foundation of the *complex symplectic geometry* of Lagrangian submanifolds in a hyperkähler manifold and studied the Legendre transformation in this setting. Unexpectedly, I discovered a beautiful formula for intersection numbers of Lagrangian subvarieties under the Legendre transformation. This formula generalizes the classical *Plücker formula* for dual plane curves to higher dimensions. The proof of this formula uses the intersection theory I developed in the paper and an equivalence of two derived categories.

In [6], I will show that the Fourier transformation on hyperkähler manifolds takes the complex symplectic geometry to the hypercomplex geometry, and I will explain some intriguing links between this transformation and the geometry of K3 surfaces.

G_2-manifolds have richest structures as they are special $\mathbb{O}$-manifolds [1] and the octonion $\mathbb{O}$ is the largest normed division algebra. In [7], Lee and I define and study geometric structures on various moduli spaces associated to G_2-manifolds. For example, the *Yukawa cubic forms* gives crucial informations about degenerations of G_2-structures. We also discuss the mirror duality for G_2-manifolds and give partial verifications.

In [8], I propose a *Topological Quantum Field Theory* for G_2-manifolds and Calabi-Yau threefolds. This is analogous to the Donaldson-Floer theory for manifolds of dimension four and three and it is defined by counting Anti-Self-Dual connections over coassociative submanifolds in G_2-manifolds.

I am also studying the *interestion theory* of coassociative submanifolds in G_2-manifolds. On the one hand, this generalizes Floer's Lagrangian intersection theory. On the other hand, this globalize Taubes' theory on pseudo-holomorphic curves on four manifolds with degenerated symplectic forms.

2. Gromov-Witten invariants for Calabi-Yaus

In my joint work with J. Bryan, [10, 11, 12], we proved the *Göttsche-Yau-Zaslow conjecture* for primitive classes in K3 surfaces and Abelian surfaces. These formulas express the generating functions for the number of curves in terms of well-known quasi-modular forms. Our approach is to first reduce the problem to computations of family Gromov-Witten invariants on elliptically fibered surfaces. Because of the special geometry of the fibration, we can express the invariants as a product of local contributions. In order to calculate these local contributions, we developed a *matching technique* to relate a difficult obstruction bundle problem to a series of other obstruction bundle problems that could be solved by other geometric means. This matching technique was later used again in the paper [13] by Bryan, Katz and myself in proving multiple covered formula for Gromov-Witten invariants of singular curves in a Calabi-Yau threefold.

3. Gieseker stability and Hermitian Yang-Mills bundles

I found the differential geometric analog of Gieseker stability in my thesis [14]. I proved that there is a correspondence between *Gieseker stable bundles* and Hermitian bundles admitting deformed Hermitian-Yang-Mills connections. The linearized version of this correspondence is the theorem of Donaldson, Uhlenbeck and Yau on the equivalence between the Mumford stability and the Hermitian-Yang-Mills bundles.

My approach in solving this fully nonlinear system of elliptic equations is to use the singular perturbation method. To apply it, I need to analyze the many layers of obstructions to perturbations in a very careful manner and I managed to identify them with different terms in the Hilbert polynomials for each bundle that occurs in the Jordan-Hölder filtration of a Gieseker stable bundle.

This deformed Hermitian-Yang-Mills equation fits nicely into an infinite dimensional moment maps and symplectic quotients picture as explained in [15]. Similar equations for Kähler-Einstein manifolds are studied in [22], they also arise as equations for supersymmetric cycles in string theory and they are mirror objects to special Lagrangian submanifolds [4].

4. Miscellaneous

In [16] and [17], I studied the *Seiberg-Witten* equations and derived Chern number inequalities for Einstein four manifolds. Using them, I obtained results on *uniformizations* of locally symmetric four manifolds of non-compact type in terms of Seiberg-Witten invariants and characteristic numbers.

The paper [18] with V. Reiner studied the *signature of toric varieties* and we applied our result to prove a combinatorial version of the *Hopf conjecture* on the Euler characteristic of non-positively curved spaces.

In [19], I explained beautiful links between G-bundles on rational surfaces and configurations of lines and rulings. In [21], Bryan, Donagi and I discussed the hyperkähler aspect of moduli spaces of G-bundles on Abelian surfaces.

I also studied various aspects of Kähler-Einstein metrics in [22, 23].

In [24], Wan and I generalized the *Witten-Yau theorem* for conformally compact manifolds to the harmonic maps setting.

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