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ADE-bundle over rational surfaces, configuration of lines and rulings

Naichung Conan Leung*
School of Mathematics,
University of Minnesota,
Minneapolis, MN 55455, U.S.A..
and
IHES,
91440 Bures-Sur-Yvette, France.

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Abstract

To each del Pezzo surface X_n (resp. ruled surface, ruled surface with a section), we describe a natural Lie algebra bundle of type E_n (resp. D_{n-1}, A_{n-2}) over it .

Using lines and rulings on any such surface, we describe various representation bundles corresponding to fundamental representations of the corresponding Lie algebra.

When we specify a geometric structure on the surface to reduce the Lie algebra to a smaller one, then the classical geometry of the configuration of lines and rulings is encoded beautifully by the branching rules in Lie theory. We discuss this relationship in details.

When we degenerate the surface to a non-normal del Pezzo surface, we discover that the configurations of lines and rulings are also governed by certain branching rules. However the degeneration theory of the bundle is not fully understood yet.

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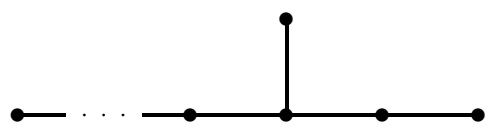
1 Introduction and results

[1]**Lines on del Pezzo surfaces**

Studying of lines on a cubic surface is a fascinating subject and it has a very long history [Segre]. It is a classical fact that there are 27 lines on a cubic surface. Moreover the configuration of these lines possesses a high degree of symmetry which is closely related to the Weyl group of E_6. Similar results hold true for other del Pezzo surfaces (see e.g. [Manin]). Recall that a del Pezzo surface is a smooth surface with an ample anti-canonical divisor. Such a surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of $\mathbb{P}^2$ at n generic points with $n \leq 8$.We call the latter surface X_n. When $n \leq 6$ the anti-canonical linear system embeds X_n in $\mathbb{P}^d$ as a degree d surface and $d = 9 - n$. A line on X_n is then equivalent to a divisor l with $l^2 = l \cdot K = -1$. We will continue to call such divisor a line even $n = 7$ or 8. Analogous to the cubic surface case, lines on X_n possess a high degree of symmetry which is closely related to the Weyl group of E_n.



$: E_n$

When $n \leq 5$, E_n coincides with classical Lie algebras as follows, $E_5 = \mathbf{so}\,(10)$, $E_4 = \mathbf{sl}\,(5)$, $E_3 = \mathbf{sl}\,(3) \times \mathbf{sl}\,(2)$, $E_2 = \mathbf{sl}\,(2) \times \mathbf{u}\,(1)$ and $E_1 = \mathbf{u}\,(1)$ (or $\mathbf{sl}\,(2)$ which we denote $\overline{E}_1$).

If we look at the fundamental representation of the Lie algebra E_n corresponding the left end node of its Dynkin diagram and call it $\mathbf{L}_n$. Then we see an interesting relationship between $\dim \mathbf{L}_n$ and the number of lines on X_n :

n	1	2	3	4	5	6	7	8
$\dim \mathbf{L}_n$	1	3	6	10	16	27	56	248
# lines on X_n	1	3	6	10	16	27	56	240

Namely $\dim \mathbf{L}_n$ equals the number of lines on X_n except when $n = 8$. In that exceptional case $\mathbf{L}_8$ coincides with the adjoint representation of E_8 and $\dim \mathbf{L}_8$ equals the number of lines on X_8 plus the rank of $E_8 = 8$. Motivated from this

[1]The content of this paper overlaps partially with a recent preprint of Friedman and Morgan 'Exceptional groups and del Pezzo surfaces' (math.AG/0009155), they construct E_n bundles over del Pezzo surfaces, possibly with rational double point singularities. They also discuss reducing the structure groups. We pay more emphasis on the interplays between representation theory of exceptional Lie algebra, reduction of these bundles and classical geometry of configurations of lines and rulings on rational surfaces. We also learn from their preprint that our bundles are really conformal bundles as in their paper.

we define a holomorphic bundle $\mathcal{L}_n$ on X_n using its lines as follows:

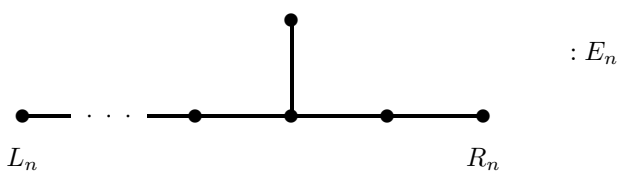
$$\begin{aligned}\mathcal{L}_n &= \bigoplus_{l:\text{line on } X_n} O(l) \qquad \text{when } n \leq 7,\\ \mathcal{L}_8 &= \bigoplus_{l:\text{line on } X_8} O(l) + O(-K)^{\oplus 8}.\end{aligned}$$

We will show that E_n is the structure group for $\mathcal{L}_n$. Moreover different specializations of X_n would reduce its structure group to various subgroups of E_n. On these specializations of X_n, special configurations of lines would translate into branching rules for the representation $\mathbf{L}_n$ under various subgroups of E_n.

Rulings on del Pezzo surfaces

Besides lines, another geometric structure on X_n is a ruling on X_n, which is a fibration of X_n over $\mathbb{P}^1$ whose generic fiber is a smooth rational curve. Any fiber R of a ruling satisfies $R^2 = 0$ and $R \cdot K = -2$. Conversely given any such divisor, it is the fiber class of a unique ruling on X_n. For example there are two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$ and one ruling on X_1, the blowup of $\mathbb{P}^2$ at a point. As we will see, these two cases correspond to standard representations of $\overline{E}_1 = \mathbf{sl}(2)$ and $E_1 = \mathbf{u}(1)$ respectively.

If we look at the fundamental representation of the Lie algebra E_n corresponding to the right end node of its Dynkin diagram and call it $\mathbf{R}_n$.



$: E_n$

Then we see an interesting relationship between $\dim \mathbf{R}_n$ and the number of rulings on X_n :

n	1	2	3	4	5	6	7
$\dim \mathbf{R}_n$	1	2	3	5	10	27	133
# rulings on X_n	1	2	3	5	10	27	126

Namely $\dim \mathbf{R}_n$ equals the number of rulings on X_n for $n \leq 6$. When $n = 7$, $\mathbf{R}_7$ coincides with the adjoint representation of E_7 and $\dim \mathbf{R}_7$ equals the number of rulings on X_7 plus the rank of E_7, which is 7. Motivated from this we define a holomorphic bundle $\mathcal{R}_n$ on X_n using its rulings as follows:

$$\begin{aligned}\mathcal{R}_n &= \bigoplus_{R:\text{ruling on } X_n} O(R) \qquad \text{when } n \leq 6,\\ \mathcal{R}_7 &= \bigoplus_{R:\text{ruling on } X_7} O(R) + O(-K)^{\oplus 7}.\end{aligned}$$

The construction of $\mathcal{R}_8$ includes more than rulings on X_8 and we shall not discuss it here. Again we will see that the structure group for $\mathcal{R}_n$ is given by E_n. Different specializations of X_n would reduce its structure group to various subgroups of E_n. On these specializations of X_n, special configurations of rulings would translate into branching rules for the representation $\mathbf{R}_n$ under various subgroups of E_n.

Cubic surfaces

For example in the cubic surface X_6 case, if we fix any line L on it, then there are 10 other lines intersecting L and 16 other lines disjoint from L. In terms of $\mathcal{L}_6$, the choice of a line reduces the structure group from E_6 to E_5 and we have a decomposition of representation bundles.

$$\mathcal{L}_6 = \pi^* \mathcal{L}_5 + \pi^* \mathcal{R}_5 \otimes O(-L) + O(L),$$

where $\pi : X_6 \to X_5$ is the blowdown of L. This decomposition corresponds to the branching rule $\mathbf{L}_6|_{E_5} = \mathbf{L}_5 + \mathbf{R}_5 + \mathbf{1}$, where $\mathbf{1}$ denotes the trivial representation. The component $\pi^* \mathcal{L}_5$ corresponds to pullback of lines from X_5 and hence disjoint from L. Each direct summand of the component of $\pi^* \mathcal{R}_5 \otimes O(-L)$ corresponds to the pullback of a particular member of a ruling which passes through the blowup point. Therefore such lines on X_6 would intersect L at one point. Moreover these 10 lines intersecting L divides into 5 pairs, each pair forms a triangle with L. This corresponds to the fact the E_5 equals $\mathbf{so}(10)$ and $\mathbf{R}_5$ is the standard representation of $\mathbf{so}(10)$.

In general, if we fix a ruling on X_n then the structure group of $\mathcal{L}_n$ or $\mathcal{R}_n$ would reduce to $D_{n-1} = \mathbf{so}(2n-2)$. If we further choose a line section of this ruling then the structure group would further reduce to $A_{n-2} = \mathbf{sl}(n-1)$. various geometric properties of lines and rulings on X_n related to corresponding branching rules will be discussed in section three and four. One should notice that we can discuss D_{n-1}-bundle (resp. A_{n-2}-bundle) on blowup of $\mathbb{P}^2$ at n generic points together with a ruling (resp. a ruling with a section) without any restriction on n.

For rulings on a cubic surface, they are in one-to-one correspondence with lines. Namely, given any ruling on X_6, there is a unique line which is a 2-section of the ruling. This is because $\mathcal{R}_6 = \mathcal{L}_6^* \otimes O(-K)$ which reflects the isomorphism $\mathbf{R}_6 = \mathbf{L}_6^*$ between E_6-representations.

When we degenerate the cubic surface to a union of three planes, then the structure Lie algebra would reduce from E_6 to $\mathbf{sl}(3) \times \mathbf{sl}(3) \times \mathbf{sl}(3)$. When we degenerate it to a union of a hyperplane and a smooth quadratic surface, then the structure Lie algebra would reduce from E_6 to $\mathbf{sl}(2) \times \mathbf{sl}(6)$. Details of these cases will be given in section six. These two are examples of degenerations into nonnormal del Pezzo surfaces [Reid1], the discussions on exactly how $\mathcal{L}_n$ and $\mathcal{R}_n$ degenerate in such situations is not completed here. We hope to come back to this problem in the future.

In general $\mathcal{L}_n$ and $\mathcal{R}_n$ are representation bundles of a E_n-bundle $\mathcal{E}_n$ over X_n

which we now describe.

(i) $\mathcal{E}_4$	is the automorphism bundle of $\mathcal{R}_4$ preserving $\Lambda^5\mathcal{R}_4 \cong O(-2K)$.
(ii) $\mathcal{E}_5$	is the automorphism bundle of $\mathcal{R}_5$ preserving $q_5 : \mathcal{R}_5 \otimes \mathcal{R}_5 \to O(-K)$.
(iii) $\mathcal{E}_6$	is the automorphism bundle of $\mathcal{R}_6$ and $\mathcal{L}_6$ preserving $c_6 : \mathcal{L}_6 \otimes \mathcal{L}_6 \to \mathcal{R}_6$, and $c_6^* : \mathcal{R}_6 \otimes \mathcal{R}_6 \to \mathcal{L}_6 \otimes O(-K)$.
(iv) $\mathcal{E}_7$	is the automorphism bundle of $\mathcal{L}_7$ preserving $f_7 : \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \to O(-2K)$.
(v) $\mathcal{E}_8$	is the automorphism bundle of $\mathcal{L}_8$ preserving $\mathcal{L}_8 \wedge \mathcal{L}_8 \to \mathcal{L}_8 \otimes O(-K)$.

Physics motivations

These E_n-bundles are related to F-theory in physics. They are also defined and studied in a very recent paper by Friedman and Morgan [FM]. If Σ is an anti-canonical curve in X_n, then Σ is of genus one. Restricting $\mathcal{E}_n$ to Σ for various complex structures on X_n and various embeddings of Σ into X_n gives the moduli space of flat E_n-bundles over the elliptic curve Σ (see [Donagi] and [FMW]). Reversing this process one would construct a degenerated K3 surface from each $E_8 \times E_8$ flat bundle over Σ. To globalize this construction, given a stable $E_8 \times E_8$-bundle over an elliptically fibered Calabi-Yau threefold, we expect to obtain a degenerated Calabi-Yau fourfold with a K3 fibration. Duality from physics predicts that certain string theory or F-theory on these spaces are equivalent to one another.

On the other hand, if X_n is embedded in a Calabi-Yau threefold M and we choose a family of Ricci flat metrics on M so that the size of X_n goes to zero (i.e. blowing down X_n inside M). Then M-theory duality predicts that enhanced gauge symmetry exists in the limit. This phenomenon might be closely related to our bundle $\mathcal{E}_n$, $\mathcal{L}_n$ and $\mathcal{R}_n$ and their various reductions.

Now we come back to the focus of this paper. To discuss the reduction of $\mathcal{E}_n$ into various smaller subgroups, it is easier to deal with the associated Lie algebra bundle $L\mathcal{E}_n$ which equals $L\mathcal{E}_n = O_X^{\oplus n} \bigoplus_D O_X(D)$ where the summation is over those divisors D satisfying $D^2 = -2$ and $D \cdot K = 0$. Throughout the paper, we discuss several ways to construct the fiberwise Lie algebra structure on $O_X^{\oplus n} \bigoplus_D O_X(D)$.

For the rest of this introduction, we will describe decompositions of $L\mathcal{E}_n$ and its representation bundles associated to various specializations or degenerations of X_n. Readers should consult individual sections for discussions of their geometric meanings. First we consider the situation where we blow down a line L on X_n. We write the blowdown morphism as $\pi : X_n \to X_{n-1}$. Then we have the following results.

Theorem 1 *If L is a line on X_n, then we have the following decompositions as representation bundles of $\pi^* L\mathcal{E}_{n-1}$ (or $\pi^* L\mathcal{E}_7 + L\mathcal{A}_1$ when $n = 8$).*

$$\begin{aligned} L\mathcal{E}_n &= \pi^* L\mathcal{E}_{n-1} + O_{X_n} + \pi^* \mathcal{L}_{n-1} \otimes O(-L) + \pi^* \mathcal{L}_{n-1}^* \otimes O(L), \text{ for } n \leq 7, \\ L\mathcal{E}_8 &= \pi^* L\mathcal{E}_7 + L\mathcal{A}_1 + \pi^* \mathcal{L}_7 \otimes O(-L) \otimes \Lambda_1. \end{aligned}$$

Here $\mathcal{A}_1$ is the automorphism bundle of $\Lambda_1 = O + O(L + K)$ preserving its determinant.

For $\mathcal{L}_n$ we have

$$\begin{aligned} \mathcal{L}_n &= \pi^* \mathcal{L}_{n-1} && + \pi^* \mathcal{R}_{n-1} \otimes O(-L) + O(L), \text{ for } n \leq 6, \\ \mathcal{L}_7 &= \pi^* \mathcal{L}_6 && + \pi^* \mathcal{R}_6 \otimes O(-L) + O(L) + O(-K - L), \\ \mathcal{L}_8 &= \pi^* \mathcal{L}_7 \otimes \Lambda_1^* && + \pi^* \mathcal{R}_7 \otimes O(-L) + \mathcal{A}_1 \otimes O(-K). \end{aligned}$$

These decompositions describe the configuration of lines on X_n with respect to the fixed line L.

In particular, if we restrict our attention to a fiber over any point on X_n, we recover the following Lie algebra facts:

Remark 2 *E_{n-1} is a Lie subalgebra of E_n for $n \leq 7$ whose Dynkin diagram is obtained by removing the node in the Dynkin diagram of E_n which corresponds to the fundamental representation $\mathbf{L}_n$. When $n = 8$, E_8 has a Lie subalgebra $E_7 + A_1$. As a representation of E_{n-1} (or $E_7 + A_1$), we have the following decomposition of E_n.*

$$\begin{aligned} E_n &= E_{n-1} + \mathbf{1} + \mathbf{L}_{n-1} + \mathbf{L}_{n-1}^*, \text{ for } n \leq 7, \\ E_8 &= E_7 + A_1 + \mathbf{L}_7 \otimes \Lambda_1. \end{aligned}$$

For $\mathbf{L}_n$, we have

$$\begin{aligned} \mathbf{L}_n &= \mathbf{L}_{n-1} + \mathbf{R}_{n-1} + \mathbf{1}, \text{for } n \leq 6, \\ \mathbf{L}_7 &= \mathbf{L}_6 + \mathbf{R}_6 + \mathbf{1} + \mathbf{1}, \\ \mathbf{L}_8 &= \mathbf{L}_7 \otimes \Lambda_1 + \mathbf{R}_7 + A_1. \end{aligned}$$

Second we consider the situation where we specify a ruling R on X_n. In this case we can consider blowup of $\mathbb{P}^2$ at an arbitrary number of points. That is there is no restriction on n for our surface X_n.We have the following results.

Theorem 3 *(1) If R determines a ruling on X_n, then the rank $2n-2$ vector bundle*

$$\mathcal{W}_{n-1} = \bigoplus_{\substack{C^2=-1\\ CK=-1\\ CR=0}} O_X(C),$$

carries a fiberwise non-degenerate quadratic form

$$q_{n-1} : \mathcal{W}_{n-1} \otimes \mathcal{W}_{n-1} \to O_{X_n}(R).$$

The automorphism bundle of $\mathcal{W}_{n-1}$ preserving q_{n-1} is a D_{n-1}-bundle $\mathcal{D}_{n-1}$ over X_n whose associated Lie algebra bundle equals

$$L\mathcal{D}_{n-1} = \Lambda^2\mathcal{W}_{n-1} \otimes O_{X_n}(-R) = O_X^{\oplus n-1} \bigoplus_{\substack{C^2=-2\\ CK=0\\ CR=0}} O_X(C).$$

The representation bundle of $L\mathcal{D}_{n-1}$ corresponding to spinor representations are

$$\mathcal{S}^+ = \bigoplus_{\substack{S^2=-1\\ SK=-1\\ SR=1}} O_{X_n}(S) \quad \text{and} \quad \mathcal{S}^- = \bigoplus_{\substack{T^2=-2\\ TK=0\\ TR=1}} O_{X_n}(T).$$

They are related by Clifford multiplication homomorphisms

$$\mathcal{S}^+ \otimes \mathcal{W}_{n-1}^* \to \mathcal{S}^- \text{ and } \mathcal{S}^- \otimes \mathcal{W}_{n-1} \to \mathcal{S}^+ ,$$

When $n = 2m$ is even, we have isomorphism

$$\left(\mathcal{S}^+\right)^* \otimes O_{X_{2m}}((m-4)R - K_X) \cong \mathcal{S}^-.$$

and when $n = 2m+1$ is odd, we have isomorphisms

$$\begin{aligned} \left(\mathcal{S}^+\right)^* \otimes O_{X_{2m+1}}((m-3)R - K_X) &\cong \mathcal{S}^+, \\ \left(\mathcal{S}^-\right)^* \otimes O_{X_{2m+1}}((m-4)R - K_X) &\cong \mathcal{S}^-. \end{aligned}$$

(2) When $n \leq 8$ then $L\mathcal{D}_{n-1}$ is a Lie algebra subbundle of $L\mathcal{E}_n$. For $n = 7$ in fact $L\mathcal{E}_7$ has a Lie algebra subbundle $L\mathcal{D}_6 + L\mathcal{A}_1$ and when $n = 8$, $L\mathcal{E}_8$ has a Lie algebra subbundle $L\mathcal{D}_8$. We can decompose representation bundles of $L\mathcal{E}_n$ under $L\mathcal{D}_{n-1}$ as follows: For $\mathcal{L}_n$ we have

$$\begin{aligned} \mathcal{L}_n &= \mathcal{W}_{n-1} + \mathcal{S}^+ \text{ when } n \leq 5, \\ \mathcal{L}_6 &= \mathcal{W}_5 + \mathcal{S}^+ + O_{X_6}(-K_X - R), \\ \mathcal{L}_7 &= \mathcal{W}_6 \otimes \Lambda_1 + \mathcal{S}^+, \\ \mathcal{L}_8 &= \mathcal{W}_8 + \mathcal{S}^+. \end{aligned}$$

This describes the relationship between the ruling and configuration of lines on X_n. For $\mathcal{R}_n$ we have

$$\begin{aligned}
\mathcal{R}_n &= O(R)\left(O+\mathcal{S}^-\right), \quad \text{for } n \leq 4, \\
\mathcal{R}_5 &= O(R)\left(O+\mathcal{S}^- + O(-K-2R)\right), \\
\mathcal{R}_6 &= O(R)\left(O+\mathcal{S}^- + \mathcal{W}_5 \otimes O(-K-2R)\right), \\
\mathcal{R}_7 &= O(R)\left(S^2\Lambda_1 + \mathcal{S}^- \otimes \Lambda_1 + \Lambda^2\mathcal{W}_6 \otimes (-K-2R)\right).
\end{aligned}$$

This describes the configuration of ruling on X_n with respect to a fixed ruling. For $L\mathcal{E}_n$ we have

$$\begin{aligned}
L\mathcal{E}_n &= L\mathcal{D}_{n-1} + O + \mathcal{S}^- + \mathcal{S}^- \otimes O((4-m)R+K), \quad \text{for } n = 2m+1 \leq 5, \\
L\mathcal{E}_n &= L\mathcal{D}_{n-1} + O + \mathcal{S}^- + \mathcal{S}^+ \otimes O((4-m)R+K), \quad \text{for } n = 2m \leq 6, \\
L\mathcal{E}_7 &= L\mathcal{D}_7 + L\mathcal{A}_1 + \mathcal{S}^- \otimes \Lambda_1 \otimes O(R+K), \\
L\mathcal{E}_8 &= L\mathcal{D}_8 + \mathcal{S}^+.
\end{aligned}$$

In particular, if we restrict our attention to a fiber over any point on X_n, we recover the following Lie algebra facts:

Remark 4 *(1) The space of infinitesimal automorphisms of a non-degenerate quadratic form on $W_{n-1} \cong \mathbb{C}^{n-1}$ is a Lie algebra of type $D_{n-1} = \mathbf{so}(2n-2)$. We can identify this Lie algebra with $\Lambda^2 W_{n-1}$. There are Clifford multiplication homomorphisms between the two spinor representation of D_{n-1} :*

$$S^+ \otimes W_{n-1}^* \to S^- \text{ and } S^- \otimes W_{n-1} \to S^+ .$$

When $n = 2m$ is even, we have isomorphism

$$\left(S^+\right)^* \cong S^-.$$

and when $n = 2m+1$ is odd, we have isomorphisms

$$\left(S^+\right)^* \cong S^+ \text{ and } \left(S^-\right)^* \cong S^-.$$

(2) When $n \leq 8$ then D_{n-1} is a Lie subalgebra of E_n whose Dynkin diagram is obtained by removing the node in the Dynkin diagram of E_n which corresponds to the fundamental representation $\mathbf{R}_n$. For $n = 7$ in fact E_7 has a Lie subalgebra $D_6 + A_1$ and when $n = 8$, E_8 has a Lie subalgebra D_8. We can decompose representations of E_n under D_{n-1} as follows: For $\mathbf{L}_n$ we have

$$\begin{aligned}
\mathbf{L}_n &= W_{n-1} + S^+ \text{ when } n \leq 5, \\
\mathbf{L}_6 &= W_5 + S^+ + \mathbf{1}, \\
\mathbf{L}_7 &= W_6 \otimes \Lambda_1 + S^+, \\
\mathbf{L}_8 &= W_8 + S^+.
\end{aligned}$$

For $\mathbf{R}_n$ *we have*

$$\begin{aligned}
\mathbf{R}_n &= \mathbf{1} + S^-, \quad \textit{for } n \leq 4 \\
\mathbf{R}_5 &= \mathbf{1} + S^- + \mathbf{1}, \\
\mathbf{R}_6 &= \mathbf{1} + S^- + W_5, \\
\mathbf{R}_7 &= \mathbf{1} \otimes S^2\Lambda_1 + S^- \otimes \Lambda_1 + \Lambda^2 W_6 \otimes \mathbf{1}.
\end{aligned}$$

For E_n *we have*

$$\begin{aligned}
E_n &= D_{n-1} + O + S^- + S^-, \quad \textit{for } n = 2m+1 \leq 5, \\
E_n &= D_{n-1} + O + S^- + S^+, \quad \textit{for } n = 2m \leq 6, \\
E_7 &= D_7 + A_1 + S^- \otimes \Lambda_1, \\
E_8 &= D_8 + S^+.
\end{aligned}$$

Third we again consider the situation where we specify a ruling R on X_n. However we also fix a line section (resp. ruling section) of this ruling. This corresponds to choosing a direct summand of $\mathcal{S}^+$ (resp. $\mathcal{S}^-$). We have the following results.

Theorem 5 *(1) If R determines a ruling on X_n and $O(S)$ is a direct summand of $\mathcal{S}^+$, then S is a line section of the ruling. We consider the rank $n-1$ bundle*

$$\Lambda_{n-2} = \bigoplus_{\substack{C^2=-1\\ CK=-1\\ CR=0\\ CS=1}} O_{X_n}(C),$$

which has determinant equals $O_{X_n}(-K_X - 2S + (n-4)R)$. The automorphism bundle of Λ_{n-2} preserving its determinant is a A_{n-2}-bundle $\mathcal{A}_{n-2}$ over X_n. Then $\mathcal{A}_{n-2}$ is a principal subbundle of $\mathcal{D}_{n-1}$. If Λ_{n-2}^l denotes the lth exterior power of Λ_{n-2}. We have the following decompositions of representation bundles of $L\mathcal{D}_{n-1}$ under $L\mathcal{A}_{n-2}$.

$$\begin{aligned}
L\mathcal{D}_{n-1} &= L\mathcal{A}_{n-2} + O + \Lambda_{n-2}^2 \otimes O(-R) + \left(\Lambda_{n-2}^2\right)^* \otimes O(R),\\
\mathcal{W}_{n-1} &= \Lambda_{n-2} + \Lambda_{n-2}^* \otimes O_{X_n}(R)\\
&= \Lambda_{n-2} + \Lambda_{n-2}^{n-2} \otimes O(K + 2S + (5-n)R),\\
\mathcal{S}^+ &= \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \Lambda_{n-2}^{2l} \otimes O(S - lR),\\
\mathcal{S}^- &= \sum_{l=1}^{\left[\frac{n}{2}\right]} \Lambda_{n-2}^{2l-1} \otimes O(S - lR).
\end{aligned}$$

When $n \le 7$, these decompositions describe the configuration of lines and rulings on X_n in relation to the ruling R with a line section S.

(2) If R determines a ruling on X_n and $O(T)$ is a direct summand of $\mathcal{S}^-$, then $R+T$ is another ruling on X_n and it is also a section of the original ruling R. We consider the rank $n-1$ bundle

$$\bar{\Lambda}_{n-2} = \bigoplus_{\substack{C^2=-1\\ CK=-1\\ CR=0\\ CT=1}} O_{X_n}(C)$$

which has determinant equals $O_{X_n}(-K_X - 2T + (n-5)R)$. The automorphism bundle of $\bar{\Lambda}_{n-2}$ preserving its determinant is a principal A_{n-2}-bundle $\bar{\mathcal{A}}_{n-2}$ over X_n. Then $\bar{\mathcal{A}}_{n-2}$ is a principal subbundle of $\mathcal{D}_{n-1}$. If $\bar{\Lambda}_{n-2}^l$ denotes the lth exterior power of $\bar{\Lambda}_{n-2}$. We have the following decompositions of

representation bundles of $L\mathcal{D}_{n-1}$ under $L\bar{\mathcal{A}}_{n-2}$.

$$\begin{aligned}
L\mathcal{D}_{n-1} &= L\bar{\mathcal{A}}_{n-2} + O + \bar{\Lambda}^2_{n-2} \otimes O(-R) + \left(\bar{\Lambda}^2_{n-2}\right)^* \otimes O(R), \\
\mathcal{W}_{n-1} &= \bar{\Lambda}_{n-2} + \bar{\Lambda}^*_{n-2} \otimes O_{X_n}(R) \\
&= \bar{\Lambda}_{n-2} + \bar{\Lambda}^{n-2}_{n-2} \otimes O(K + 2T + (6-n)R) \\
\mathcal{S}^+ &= \sum_{l=1}^{\left[\frac{n}{2}\right]} \bar{\Lambda}^{2l-1}_{n-2} \otimes O(T - (l-1)R). \\
\mathcal{S}^- &= \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \bar{\Lambda}^{2l}_{n-2} \otimes O(T - lR).
\end{aligned}$$

When $n \leq 7$, these decompositions describe the configuration of lines and rulings on X_n in relation to the rulings R and $R+T$.

In particular, if we restrict our attention to a fiber over any point on X_n, we recover the following Lie algebra facts:

Remark 6 *(1) $A_{n-2} = \mathbf{sl}(n-1)$ is a Lie subalgebra of $D_{n-1} = \mathbf{so}(2n-2)$ whose Dynkin diagram is obtained by removing the node in the Dynkin diagram of D_{n-1} which corresponds to the fundamental representation S^+. We have the following decomposition of D_{n-1} representations under A_{n-2}*

$$\begin{aligned}
D_{n-1} &= A_{n-2} + \mathbf{1} + \Lambda^2_{n-2} + \left(\Lambda^2_{n-2}\right)^*, \\
W_{n-1} &= \Lambda_{n-2} + \Lambda^*_{n-2} = \Lambda_{n-2} + \Lambda^{n-2}_{n-2}, \\
S^+ &= \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \Lambda^{2l}_{n-2} \text{ and } S^- = \sum_{l=1}^{\left[\frac{n}{2}\right]} \Lambda^{2l-1}_{n-2}.
\end{aligned}$$

Here Λ^l_{n-2} denotes the lth exterior power of the standard representation of A_{n-2}.

(2) A_{n-2} is also a Lie subalgebra of D_{n-1} whose Dynkin diagram is obtained by removing the node in the Dynkin diagram of D_{n-1} which corresponds to the fundamental representation S^-. We have the following decomposition of D_{n-1} representations under A_{n-2}

$$\begin{aligned}
D_{n-1} &= A_{n-2} + \mathbf{1} + \Lambda^2_{n-2} + \left(\Lambda^2_{n-2}\right)^*, \\
W_{n-1} &= \Lambda_{n-2} + \Lambda^*_{n-2} = \Lambda_{n-2} + \Lambda^{n-2}_{n-2}, \\
S^+ &= \sum_{l=1}^{\left[\frac{n}{2}\right]} \Lambda^{2l-1}_{n-2} \text{ and } S^- = \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \Lambda^{2l}_{n-2}.
\end{aligned}$$

Fourth we consider degenerating X_n to a nonnormal del Pezzo surface. We have the following result.

Theorem 7 *We consider a family of smooth del Pezzo surfaces $X_n(t)$ degenerating to a nonnormal surface $X_n(0)$. Let Z be the limit of the intersection of $X_n(t)$ and the singular locus of $X_n(0)$ as t goes to zero. We write $\mathcal{I} = \bigoplus_{q\in Z} \mathcal{I}_{\{q\}}$.*

(1) When $n = 5$ and we degenerate the quartic surface X_5 inside $\mathbb{P}^4$ into a union of two quadratic surfaces $X_5(0) = Q_{(1)} \cup Q_{(2)}$. Let $\mathcal{R}_{(i)} = \bigoplus_R O_{Q_{(i)}}(R)$ with R satisfying $R^2 = 0$ and $R \cdot K_{Q_{(i)}} = -2$, then we write

$$\begin{aligned} \mathcal{L}_5(0) &= \mathcal{I} \otimes \mathcal{R}_{(1)} + \mathcal{I} \otimes \mathcal{R}_{(1)}, \\ \mathcal{R}_5(0) &= \mathcal{I} \wedge \mathcal{I} + \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}. \end{aligned}$$

These decomposition describe the limit configurations of lines and rulings on $X_5(t)$ as t approach zero.

(2) When $n = 6$ and we degenerate the cubic surface X_6 into a union of three planes $X_6(0) = H_1 \cup H_2 \cup H_3$. Then we write

$$\mathcal{L}_6(0) = \mathcal{I}_{(1)} \otimes \mathcal{I}_{(2)} \otimes O_{H_3}(1) + \mathcal{I}_{(2)} \otimes \mathcal{I}_{(3)} \otimes O_{H_1}(1) + \mathcal{I}_{(3)} \otimes \mathcal{I}_{(1)} \otimes O_{H_2}(1).$$

This decomposition describes the limit configuration of lines on $X_6(t)$ as t approach zero. Moreover the structure of the triple product $c_6(0) : \mathcal{L}_6(0) \otimes \mathcal{L}_6(0) \otimes \mathcal{L}_6(0) \to O_{X_6(0)}(1)$ can also be described in very explicit terms. Similar results hold true for $\mathcal{R}_6$ via $\mathcal{R}_6 = \mathcal{L}_6^* \otimes O(-K)$.

(3) When $n = 6$ and we degenerate the cubic surface X_6 into a union of a plane and a quadratic surface $X_6(0) = H \cup Q$. Let $\mathcal{R}_Q = \oplus_R O_Q(R)$ for R satisfying $R^2 = 0$ and $R \cdot K_Q = -2$. Then we write

$$\mathcal{L}_6(0) = \Lambda^2 \mathcal{I} \otimes O_H(1) + \mathcal{I} \otimes \mathcal{R}_Q.$$

This decomposition describes the limit configuration of lines on $X_6(t)$ as t approach zero. Moreover the structure of the triple product

$$c_6(0) : \mathcal{L}_6(0) \otimes \mathcal{L}_6(0) \otimes \mathcal{L}_6(0) \to O_{X_6(0)}(1),$$

can also be described in explicit terms. Similar results hold true for $\mathcal{R}_6$ via $\mathcal{R}_6 = \mathcal{L}_6^* \otimes O(-K)$.

(4) When $n = 7$ and we degenerate X_7 into a union of two copies of $\mathbb{P}^2$ joining along a conic curve. Then we write

$$\mathcal{L}_7(0) = \Lambda^2 \mathcal{I} \otimes O_{\mathbb{P}^2_{(1)}}(1) + \Lambda^2 \mathcal{I} \otimes O_{\mathbb{P}^2_{(2)}}(1).$$

This decomposition describes the limit configuration of lines on $X_7(t)$ as t approach zero. Moreover the structure of the quartic product

$$f_7 : \mathcal{L}_7(0) \otimes \mathcal{L}_7(0) \otimes \mathcal{L}_7(0) \otimes \mathcal{L}_7(0) \to O(-2K),$$

can also be described in explicit terms.

If we restrict our attention to fibers over any smooth point on $X_n(0)$, they are related to the following Lie algebra facts:

Remark 8 *(1) When $n=5$, $E_5=D_5=\mathbf{so}(10)$ has a Lie subalgebra $A_3\times A_1\times A_1=\mathbf{sl}(4)+\mathbf{sl}(2)+\mathbf{sl}(2)$. We have the following decomposition of E_5 representations under $A_3\times A_1\times A_1$.*

$$\begin{aligned}\mathbf{L}_5 &= \Lambda_3\otimes\Lambda_1\otimes\mathbf{1}+\Lambda_3^*\otimes\mathbf{1}\otimes\Lambda_1,\\ \mathbf{R}_5 &= \Lambda_3^2\otimes\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes\Lambda_1\otimes\Lambda_1.\end{aligned}$$

Here Λ_n denotes the standard representation of $A_n=\mathbf{sl}(n)$.

(2) When $n=6$, E_6 has a Lie subalgebra $A_2\times A_2\times A_2=\mathbf{sl}(3)+\mathbf{sl}(3)+\mathbf{sl}(3)$. We have the following decomposition of E_6 representation under $A_2\times A_2\times A_2$.

$$\mathbf{L}_6=\Lambda_2\otimes\Lambda_2^*\otimes\mathbf{1}+\mathbf{1}\otimes\Lambda_2\otimes\Lambda_2^*+\Lambda_2^*\otimes\mathbf{1}\otimes\Lambda_2.$$

Moreover the triple product $c_6:\mathbf{L}_6\otimes\mathbf{L}_6\otimes\mathbf{L}_6\to\mathbb{C}$ can be described explicitly in terms of this decomposition. Similar structure holds for $\mathbf{R}_6=\mathbf{L}_6^*$.

(3) When $n=6$, E_6 has a Lie subalgebra $A_1\times A_5=\mathbf{sl}(2)+\mathbf{sl}(6)$. We have the following decomposition of E_6 representation under $A_1\times A_5$.

$$\mathbf{L}_6=\Lambda_5^2\otimes\mathbf{1}+\Lambda_5\otimes\Lambda_2.$$

Moreover the triple product $c_6:\mathbf{L}_6\otimes\mathbf{L}_6\otimes\mathbf{L}_6\to\mathbb{C}$ can be described explicitly in terms of this decomposition. Similar structure holds for $\mathbf{R}_6=\mathbf{L}_6^*$.

(4) When $n=7$, E_7 has a Lie subalgebra $A_7=\mathbf{sl}(8)$. We have the following decomposition of E_7 representation under A_7.

$$\mathbf{L}_7=\Lambda_7^2+\Lambda_2^6.$$

Moreover the triple product $f_7:\mathbf{L}_7\otimes\mathbf{L}_7\otimes\mathbf{L}_7\otimes\mathbf{L}_7\to\mathbb{C}$ can be described explicitly in terms of this decomposition.

Notations: We will simply use X to denote X_n when there is no confusion that might occur. The canonical divisor of X_n is called K_{X_n} or simply K. If D is a divisor on X, then there is an associated line bundle $O(D)$ on X together with a rational section of $O(D)$. Such a section is canonical up to multiplication by a non-zero scalar. We implicitly fix one such section for each D. $H^i(O(D))$ denotes the ith cohomology group of the sheaf of sections of $O(D)$ whose dimension is denoted as $h^i(O(D))$. Moreover $\chi(O(D))$ denotes the Euler characteristic, namely $\chi(O(D))=\sum_{i=0}^{2}(-1)^i h^i(O(D))$. We often use $+$ to replace $\oplus$ to improve the visual effect of our equations.

Regarding notations from Lie theory, a fundamental representation means a highest weight representation whose highest weight vector is the first lattice point along an edge of the fundamental Weyl chamber. Such vectors are in one-to-one correspondence with nodes of the Dynkin diagram. Therefore we usually speak of a fundamental representation corresponding to some particular node in the Dynkin diagram. When $n\le 5$, then E_n is a classical Lie algebra and

the standard representation of it refers to its defining representation. For example the standard representation of $\mathbf{sl}(2)$ is of dimension two and the standard representation of $\mathbf{so}(2n-2)$ is of dimension $2n-2$.

Even though we use the complex number field, most arguments in this paper work equally well over any algebraically closed field of characteristic zero.

2 E_n-bundles over del Pezzo surfaces

In this section we study the bundle $\mathcal{L}_n$, $\mathcal{R}_n$ and $L\mathcal{E}_n$ over a del Pezzo surface X_n. Before we do this we first discuss some general properties of del Pezzo surfaces. Many of these properties can be found in [Beauville], [Hartshorne] and [Manin].

2.1 Geometry of del Pezzo surfaces

As we mentioned in the introduction, a del Pezzo surface has an ample anti-canonical divisor class. We can represent X_n as a blowup of $\mathbb{P}^2$ at n generic points. When $n \leq 6$ then the complete linear system of cubics through these n points defines an embedding $X_n \subset \mathbb{P}^{9-n}$. It is an anti-canonical embedding of degree $d = 9 - n$. For example X_5 is a complete intersection of two quadrics in $\mathbb{P}^4$ and X_6 is a cubic surface in $\mathbb{P}^3$. When $n = 7$ this linear system is not very ample on X_7. Instead it defines a double cover of $\mathbb{P}^2$ branched along a quartic curve. On the other hand we can embed X_7 inside the weighted projective space $\mathbb{P}(1,1,1,2)$ and X_8 inside $\mathbb{P}(1,1,2,3)$.

Suppose we fix a point p on $X_n \subset \mathbb{P}^{9-n}$. The projection from p defines a morphism from the blowup of X_n at p to $\mathbb{P}^{9-n-1}$. We denote this blowup surface as X_{n+1}. Then this morphism is the anti-canonical morphism for X_{n+1}.

Next we want to discuss lines and rulings on X_n. When $n \leq 6$ we have the anti-canonical embedding $X_n \subset \mathbb{P}^d$. A curve in X_n is a line inside $\mathbb{P}^d$ if and only if it is an exceptional curve in X_n. By abuse of notations we continue to call an exceptional curve in X_n a line even when $n = 7$ or 8. When X_n is represented as a blowup of $\mathbb{P}^2$ at generic points $p_1, ..., p_n$:

$$\pi : X_n \to \mathbb{P}^2.$$

Then we can describe (see [Manin]) all lines in X_n as follows: C is a line in X_n if and only if (i) $\pi(D) = p_i$; or (ii) $\pi(D)$ is a line passes through p_i and p_j; or (iii) $\pi(D)$ is a conic passes through five of the p_i's; or (iv) $\pi(D)$ is a cubic passes through seven of the p_i's and with one being a double point; or (v) $\pi(D)$ is a quartic passes through 8 of the p_i's and three being double points; or (vi) $\pi(D)$ is a quintic passes through 8 of the p_i's and six being double points; or (vii) $\pi(D)$ is a sextic passes through 8 of the p_i's and seven being double points and one triple point. The number of lines on each X_n is given in the table in the introduction. The following proposition (from [Manin]) is a useful numerical characterization of lines on X_n.

Proposition 9 *If D is a divisor on X_n, then there is a line linearly equivalent to D if and only if $D^2 = -1$ and $D \cdot K = -1$.*

Proof of proposition: If D is linearly equivalent to a line, then it implies that $D^2 = -1$ and $D \cdot K = -1$ by the adjunction formula. Conversely if D satisfies these two equalities, then the Riemann-Roch formula give $\chi(O(D)) = 1$. On the other hand $K - D$ is not an effective divisor because $(-K) \cdot (K - D) < 0$ and $-K$ is an ample divisor. By Serre duality, we have $h^2(O(D)) = 0$. So

$h^0(O(D))$ is at least one and we can assume that D is an effective divisor. In fact D is an irreducible divisor because $D \cdot (-K) = 1$.

The genus formula showed that D has arithmetic genus equals zero. It implies that the irreducible divisor D is in fact a smooth rational curve. Now $D^2 = -1$ implies that it is a line.

Therefore

$$\begin{aligned} \mathcal{L}_n &= \bigoplus_{\substack{l^2=-1 \\ lK=-1}} O(l) \qquad \text{when } n \leq 7, \\ \mathcal{L}_8 &= \bigoplus_{\substack{l^2=-1 \\ lK=-1}} O(l) + O(-K)^{\oplus 8}. \end{aligned}$$

The following notion about certain types of configuration of lines on X_n will be used later.

Definition 10 *If the dual graph of a configuration of d lines on X_n is a d-gon, i.e. a polygon with d edges, then we call the configuration a d-gon.*

A d-gon is called a triangle (resp. rectangle, pentagon, hexagon, septagon, octagon) if d equals three (resp. four, five, six, seven, eight).

Proposition 11 *(1) When $n \leq 5$ there is no triangle on X_n. When $n = 6$ then every triangle on X_6 is an anti-canonical divisor.*

(2) When $n \leq 4$ there is no rectangle on X_n. When $n = 5$ then every rectangle on X_5 is an anti-canonical divisor.

Proof of proposition: The proof of part (1) and (2) are essentially the same and we will only give one of them. If $C = l_1 \cup l_2 \cup l_3$ is a triangle on X_n, then $C^2 = 3$ and $C \cdot K = -3$. Therefore if $n \leq 5$ then $K^2 = 9 - n > 3$ and hence $(C \cdot K)^2 = 9 < \left(C^2\right) \cdot \left(K^2\right)$. This violates the Hodge index theorem so no triangle exists.

When $n = 6$, we have $(C \cdot K)^2 = \left(C^2\right) \cdot \left(K^2\right)$. Again by Hodge index theorem C must be linearly equivalent to a multiple of K. Because $C \cdot K = -3$, we have $C \equiv -K$. Hence the result.

In fact the same proof shows that when $n < 9 - k$ there is no k-gon on X_n. When $n = 9 - k$ then every k-gon on X_n is an anti-canonical divisor.

Corollary 12 *If l_1 and l_2 are two lines on X_n. Then $l_1 \cdot l_2 \neq 2$ when $n \leq 6$.*

If $l_1 \cdot l_2 = 2$ and $n = 7$, then l_2 is linearly equivalent to $-K - l_1$.

Next we are going to give a similar numerical characterization of rulings on X_n.

Proposition 13 *If D is a divisor on X_n, then D determines a ruling on X_n if and only if $D^2 = 0$ and $D \cdot K = -2$.*

Proof of proposition: If D determines a ruling on X, then it implies that $D^2 = 0$ and $D \cdot K = -2$ by the adjunction formula. Conversely if D satisfies these two equalities, then the Riemann-Roch formula give $\chi(O(D)) = 2$. By similar arguments as above, we can assume that D is an effective divisor with at most two irreducible components and its complete linear system is at least one dimensional. Again the genus formula showed that D has arithmetic genus equals zero.

We claim that D has no fixed component. Otherwise $D = D_0 + R$ where R is the fixed component and D_0 is the free component. Since $-K$ is ample and $-K \cdot D = 2$, we have $-K \cdot D_0 = -K \cdot R = 1$. Since $p_a(D) = 0$ and $R^2 < 0$, R must be a line on X. Using the equality

$$\begin{aligned} 0 &= D^2 \\ &= D_0^2 + 2D_0 \cdot R - 1, \end{aligned}$$

and the fact that $D_0^2 \geq 0$, we must have $D_0^2 = 1$ and $D_0 \cdot R = 0$. Now the adjunction formula implies that $p_a(D_0) = 1$. This contradicts to $p_a(D) = 0$.

This linear system defines a fibration of X whose generic fiber is an irreducible curve with $p_a(D) = 0$ (and hence a smooth rational curve). That is D determines a ruling on X. $\square$

Hence we have

$$\mathcal{R}_n = \bigoplus_{\substack{R^2=0 \\ RK=-2}} O(R) \qquad \text{when } n \leq 6,$$

We also define

$$\mathcal{R}_7 = \bigoplus_{\substack{R^2=0 \\ RK=-2}} O(R) + O(-K)^{\oplus 7}.$$

Now we want to describe a vector bundle homomorphism

$$c_n : \mathcal{L}_n \otimes \mathcal{L}_n \to \mathcal{R}_n.$$

Suppose that l_1 and l_2 are lines on X_n, then the divisor $R = l_1 + l_2$ determines a ruling on X_n if and only if l_1 and l_2 intersect at one point. It is because

$$R \cdot K = (l_1 + l_2) \cdot K = -1 - 1 = -2$$

and

$$R^2 = (l_1 + l_2)^2 = l_1^2 + l_2^2 + 2l_1 \cdot l_2 = -2 + 2l_1 \cdot l_2.$$

Therefore $R^2 = 0$ if and only if $l_1 \cdot l_2 = 1$. In this case there is an isomorphism $O_X(l_1) \otimes O_X(l_2) \xrightarrow{\simeq} O_X(R)$.Combining these isomorphisms for various intersecting pairs of lines, we obtain the bundles homomorphism

$$c_n : \mathcal{L}_n \otimes \mathcal{L}_n \to \mathcal{R}_n,$$

for $n \leq 6$. When $n = 7$ two lines l_1 and l_2 can intersect at more than one point. When this happens, $l_1 + l_2$ is an anti-canonical divisor on X_7. Combining with this we obtain the homomorphism $\mathcal{L}_7 \otimes \mathcal{L}_7 \to \mathcal{R}_7$. See later section for details. One should notice that there could be more than one pair of lines that determines the same ruling on X_n. In fact, we will show later that, each ruling on X_n comes from exactly $n-1$ pairs of lines.

2.2 Construction of $L\mathcal{E}_n$ over X_n

Next we construct the following holomorphic vector bundle $L\mathcal{E}_n$:

$$L\mathcal{E}_n = O^{\oplus n} + \bigoplus_{\substack{D^2=-2 \\ DK=0}} O(D)$$

In Chapter four of [Manin], Manin discussed properties of these divisors D and their relationship with the root system of E_n. Here we simply use them to construct a Lie algebra bundle of type E_n over X_n.

Remark 14 *As a holomorphic vector bundle over X_n, $\mathcal{L}_n$ (resp. $\mathcal{R}_n$ and $L\mathcal{E}_n$) is a direct sum of line bundles which have the same degree with respect to the anti-canonical polarization. Therefore it is a semi-stable vector bundle over X_n.*

In the following proposition we show that the above numerical criterion on a divisor class D characterizes those divisor classes which can be written as the difference of two lines on X_n.

Proposition 15 *(1) If D is a divisor on X_n with $n \leq 7$, then $D^2 = -2$ and $D \cdot K = 0$ if and only if $D \equiv l - l'$ for some lines l and l' on X_n.*

(2) If D is a divisor on X_8, then $D^2 = -2$ and $D \cdot K = 0$ if and only if $D + K$ is a line.

Proof of proposition: Part (2) follows immediately from the previous characterization of a line on X_n and $K^2 = 1$ on X_8.

Now we assume that $n \leq 7$. Since the Neron-Severi group of X_n is spanned by the canonical divisor K and the collection of lines on X_n, a divisor with zero intersection with K and every line on X_n would necessarily be numerically trivial. However $D^2 = -2$, there must be at least one line l on X_n such that $D \cdot l$ is nonzero.

Replacing D by $-D$ if necessary, we can assume that $D \cdot l > 0$. If we write $D' = D + l$ then $D'^2 = -3 + 2D \cdot l \geq -1$. Also $D' \cdot K = -1$. Applying the Hodge index theorem, we have $\left(D'^2\right)\left(K^2\right) \leq \left(D' \cdot K\right)^2 = 1$. Therefore $D'^2 = -1$, 0 or 1. However D'^2 cannot be zero since $\acute{D}'^2$ and $D' \cdot K$ have the same parity.

If $D'^2 = 1$, then the above inequality implies that $K^2 \leq 1$. This is impossible for X_n with $n \leq 7$ because $K^2_{X_n} = 9 - n$. Hence $D'^2 = -1$ and $D' \cdot K = -1$. Namely D' is a line l' and we have $D = D' - l = l' - l$. Hence we proved the proposition. $\square$

Therefore we have

$$\begin{aligned} L\mathcal{E}_n &= O_X^{\oplus n} \bigoplus_{\substack{D=l-l' \\ l,l' \text{ disjoint lines on } X}} O_X(D) \text{ when } n \leq 7. \\ L\mathcal{E}_8 &= \mathcal{L}_8 \otimes O(K). \end{aligned}$$

Now if we blow down a fixed line L on X_n then the above description permits us to relate $L\mathcal{E}_n$ to the bundle $L\mathcal{E}_{n-1}$ on the blowdown surface X_{n-1}. Let us first denote the blow down morphism as $\pi : X_n \to X_{n-1}$. Second it is obvious that $\pi^* L\mathcal{E}_{n-1} + O_{X_n}$ is a subbundle of $L\mathcal{E}_n$. Third $\mathcal{L}_{n-1}$ is constructed in terms of lines on X_{n-1} and therefore $\pi^*\mathcal{L}_{n-1} \otimes O(-L)$ and its dual are also subbundles of $L\mathcal{E}_n$. When $n \leq 7$ this exhausts the whole bundle $L\mathcal{E}_n$.

$$L\mathcal{E}_n = \pi^* L\mathcal{E}_{n-1} + O_{X_n} + \pi^*\mathcal{L}_{n-1} \otimes O(-L) + \pi^*\mathcal{L}_{n-1}^* \otimes O(L).$$

When $n = 8$ the line bundle $O(K+L)$ and its dual also lie inside $L\mathcal{E}_8$. We have

$$\begin{aligned} L\mathcal{E}_8 = \pi^* L\mathcal{E}_7 + O_{X_8} + \pi^*\mathcal{L}_7 \otimes O(-L) + \pi^*\mathcal{L}_7^* \otimes O(L) \\ + O(-K-L) + O(K+L). \end{aligned}$$

In general the maximum number of disjoint lines on X_n equals n. If we fix such a maximal collection of lines on X_n and label them $L_1, L_2, ..., L_n$. Again we denote the blow down morphism of these lines as $\pi : X_n \to \mathbb{P}^2$. Namely the exceptional locus of π is $L_1 \cup L_2 \cup ... \cup L_n$. We also denote the pullback of the hyperplane class of $\mathbb{P}^2$ by H. Then, for $n < 8$, we have

$$\begin{aligned} L\mathcal{E}_n = \bigoplus^n O_{X_n} \bigoplus_{i \neq j} O(L_i - L_j) \\ \bigoplus_{i<j<k} O(H - L_i - L_j - L_k) + O(-H + L_i + L_j + L_k) \\ \bigoplus_{i_1<...<i_6} O\left(2H - \sum_{m=1}^{6} L_{i_m}\right) + O\left(-2H + \sum_{m=1}^{6} L_{i_m}\right). \end{aligned}$$

When $n = 8$ we have

$$\begin{aligned} L\mathcal{E}_8 = \bigoplus^8 O_{X_8} \bigoplus_{i \neq j} O(L_i - L_j) \\ \bigoplus_{i<j<k} O(H - L_i - L_j - L_k) + O(-H + L_i + L_j + L_k) \\ \bigoplus_{i_1<...<i_6} O\left(2H - \sum_{m=1}^{6} L_{i_m}\right) + O\left(-2H + \sum_{m=1}^{6} L_{i_m}\right) \\ \bigoplus_{i=1}^{8} O\left(3H - \sum_{j=1}^{8} L_j - L_i\right) + O\left(-3H + \sum_{j=1}^{8} L_j + L_i\right), \end{aligned}$$

Similarly we can decompose $\mathcal{L}_n$, when $n < 8$, as

$$\mathcal{L}_n = \bigoplus_{i=1}^{n} O(L_i) \bigoplus_{i<j} O(H - L_i - L_j) \bigoplus_{i_1<\ldots<i_5} O\left(2H - \sum_{m=1}^{5} L_{i_m}\right),$$

and $\mathcal{L}_8 = L\mathcal{E}_8 \otimes O(-K)$.

2.3 Fiberwise Lie algebra structure on $L\mathcal{E}_n$

We remind the reader that the above decomposition of $L\mathcal{E}_n$ in terms of $L\mathcal{E}_{n-1}$ and $\mathcal{L}_{n-1}$ resembles the decomposition of the adjoint representation of the Lie algebra E_n under its maximal Lie subalgebra $E_{n-1}+\mathbf{u}(1)$. When $n = 8$, $E_7+\mathbf{u}(1)$ is not a maximal Lie subalgebra of E_8. In fact $E_7+\mathbf{u}(1)$ is contained in a maximal Lie subalgebra $E_7 + \mathbf{sl}(2)$. They are

$$\begin{aligned} &E_n = E_{n-1} + \mathbf{u}(1) + \mathbf{L}_{n-1} \otimes \eta^{9-n} + \overline{\mathbf{L}_{n-1} \otimes \eta^{9-n}} \quad \text{for } n \leq 7, \\ &E_8 = E_7 + A_1 + \mathbf{L}_7 \otimes \Lambda_1. \end{aligned}$$

Here η is the standard representation of $\mathbf{u}(1)$ and Λ_1 is the standard representation of $A_1 = \mathbf{sl}(2)$. This highly suggests that the bundle $L\mathcal{E}_n$ carries a fiberwise Lie algebra structure of type E_n such that $\mathcal{L}_n$ is one of its representation bundle corresponding to the representation $\mathbf{L}_n$ of E_n. There are various ways to describe this fiberwise Lie algebra structure on $L\mathcal{E}_n$. A direct approach would be to first write $L\mathcal{E}_n = \Lambda \otimes_{\mathbb{Z}} O_X \bigoplus_D O_X(D)$ where $\Lambda \subset H^2(X_n, \mathbb{Z})$ is the perpendicular complement of K_X with respect to the intersection pairing and the summation is over those D's satisfying $D^2 = -2$ and $D \cdot K = 0$. Second first Chern classes of D's are elements in Λ and they form the weight lattice of E_n (see [Manin]). This gives the bracket between $\Lambda \otimes_{\mathbb{Z}} O_X$ and $O_X(D)$. Third the bracket between elements in $O(D_1)$ and $O(D_2)$ can be described using homomorphisms $O(D_1) \otimes O(D_2) \to O(D_3)$ which equals zero unless $D_1 \cdot D_2 = 1$ and $D_3 = D_1 + D_2$, in that case, it is an isomorphism.

Instead of filling in the details of this algebraic approach, we discuss several other descriptions of the fiberwise Lie algebra structure on $L\mathcal{E}_n$ using its decomposition into subbundles. It is because these latter approaches reflect more about the geometry of lines and rulings on X_n. One way to describe the Lie algebra structure on E_8 is to break it down using its Lie subalgebra D_8, see for example [Adams]. This approach to describe the fiberwise Lie algebra structure on $L\mathcal{E}_8$ will be discussed in the last section of this paper. By restricting this bracket to $\pi^* L\mathcal{E}_7$ we obtain the fiberwise Lie algebra structure on $L\mathcal{E}_7$. Inductively we obtain the fiberwise Lie algebra structure on every $L\mathcal{E}_n$.

Two other methods to describe the Lie algebra structure on E_n when $n \leq 7$ is to break it down using its maximum Lie subalgebra D_{n-1} and E_{n-1}. We discuss using $L\mathcal{E}_{n-1}$ (resp. $L\mathcal{D}_{n-1}$) to construct the fiberwise Lie algebra structure on $L\mathcal{E}_n$ now (resp. in next section).

First let us recall that $E_n = E_{n-1} + \mathbf{u}(1) + \mathbf{L}_{n-1} \otimes \eta^{9-n} + \overline{\mathbf{L}_{n-1} \otimes \eta^{9-n}}$ when $n \leq 7$. Moreover the Lie algebra structure on E_n can be reconstruct

from E_{n-1} and $\mathbf{L}_{n-1}$ together with various pairings between them. Likewise we can construct a Lie algebra bundle of type E_n on X_n inductively from the E_{n-1}-bundle $L\mathcal{E}_{n-1}$ on X_{n-1} and the vector bundle $\mathcal{L}_{n-1}$. When $n=1$, $L\mathcal{E}_1$ is just the trivial line bundle on X_1 with the Abelian Lie algebra structure on each fiber. $\mathcal{L}_1 = O(l)$ where l is the unique line on X_1. Moreover the fiberwise action of $L\mathcal{E}_1$ on $\mathbf{L}_1$ is given by the isomorphism $O \otimes O(l) \cong O(l)$.

By induction on $n, \mathcal{E}_{n-1}$ is an E_{n-1}-bundle over X_{n-1}. This determines a fiberwise Lie bracket homomorphism

$$\alpha : L\mathcal{E}_{n-1} \otimes L\mathcal{E}_{n-1} \to L\mathcal{E}_{n-1}.$$

Moreover we have a homomorphism

$$\beta : L\mathcal{E}_{n-1} \otimes \mathcal{L}_{n-1} \to \mathcal{L}_{n-1}$$

which is induced from (i) the natural homomorphism $O(l_1 - l_2) \otimes O(l) \to O(l_1)$ which is an isomorphism when $l_2 = l$ and vanishes otherwise and (ii) the homomorphism $\Lambda \otimes O(l) \to O(l)$ which send $D \otimes s$ to $(D \cdot l)\, s$. Notice that β express $\mathcal{L}_{n-1}$ as a representation bundle of $L\mathcal{E}_{n-1}$. Similarly we have the dual representation.

$$\gamma : L\mathcal{E}_{n-1} \otimes \mathcal{L}^*_{n-1} \to \mathcal{L}^*_{n-1}$$

Furthermore we can define a homomorphism

$$\delta : \mathcal{L}_{n-1} \otimes \mathcal{L}^*_{n-1} \to L\mathcal{E}_{n-1}$$

which is induced from the natural homomorphism $O(l) \otimes O(-l') \to O(l - l')$ when l and l' are disjoint. We also have a trace homomorphism

$$t : \mathcal{L}_{n-1} \otimes \mathcal{L}^*_{n-1} \to O_X.$$

These homomorphisms defines a fiberwise Lie bracket on $\mathcal{E}_n$ using our previous decomposition as follow:

$$\begin{aligned} &[(a,b,cd,ef),(a',b',c'd',e'f')] \\ = &\begin{pmatrix} \alpha(a,a') & t(c,e)(d,f') & \beta(a,c')\,d' & \gamma(a,e')\,f \\ +\delta(c,e)(d,f') & -t(c',e)(d',f), & -\beta(a',c)\,d & -\gamma(a',e)\,f' \\ -\delta(c',e)(d',f) & & +c'b^{9-n}d' & +e'b^{9-n}f' \\ & & -c'b^{9-n}d, & -eb'^{9-n}f \end{pmatrix}. \end{aligned}$$

The fact that this bracket gives $L\mathcal{E}_n$ a fiberwise E_n Lie algebra structure over X_n follows from a corresponding construction for the Lie algebra structure on E_n [Adams]. If we want to construct the fiberwise Lie algebra structure on $L\mathcal{E}_8$ using this method, we need to suitable modify it to adopt the $\mathbf{sl}(2)$ subalgebra of E_8.

As we have mentioned in this construction that $\mathcal{L}_n$ is the representation bundle of the Lie algebra bundle $L\mathcal{E}_n$ over X_n corresponding to the fundamental

representation $\mathbf{L}_n$. Similarly the bundle $\mathcal{R}_n$ is a representation bundle of $L\mathcal{E}_n$ corresponding to the fundamental representation $\mathbf{R}_n$ of E_n. To describe its action for $n \leq 6$ we use the homomorphism $\Lambda \otimes O(R) \to O(R)$ which send $D \otimes s$ to $(D \cdot R)\, s$ and the fact that $R' = D + R$ determines a ruling on X_n if and only if $D \cdot R = 1$. In fact this happens precisely when we can write $D = l_1 - l_2$ and $R = l_2 + l_3$ such that l_1 and l_3 intersects at a point.

When $n = 7$, we suppose that $D = l_1 - l_2$ and $R = l_2 + l_3$ then l_1 and l_3 can also intersect at two points. In that case they form a 2-gon and therefore $D + R$ is linearly equivalent to $-K$. If we write

$$\mathcal{R}_7 = \Lambda \otimes_{\mathbb{Z}} O(-K) + \bigoplus_{R:\text{ruling}} O(R).$$

Then in this case the action of $s \in O(D) \subset L\mathcal{E}_7$ on $t \in O(R) \subset \mathcal{R}_7$ is given by $[D] \otimes (st) \in \Lambda \otimes_{\mathbb{Z}} O(-K) \subset \mathcal{R}_7$. Here $[D] \in \Lambda$ denotes the divisor class of D and st is an element of $O(-K)$ given by the product of s and t under the isomorphism $O(D) \otimes O(R) \cong O(-K)$.

Now if $s \in O(D) \subset L\mathcal{E}_7$ and $[D_0] \otimes u \in \Lambda \otimes_{\mathbb{Z}} O(-K) \subset \mathcal{R}_7$, then $R = D - K$ satisfies $R^2 = 0$ and $R \cdot K = -2$ and therefore R determines a ruling on X_7. The action of s on $[D_0] \otimes u$ is given by $(D \cdot D_0)(su)$ where $D \cdot D_0$ is the intersection number of D and D_0 and $su \in O(R)$ is the product of s and u under the isomorphism $O(D) \otimes O(-K) \cong O(R)$. The action of $L\mathcal{E}_7$ on $\mathcal{R}_7$ is completely determined by these products [Adams].

We remark that the homomorphism $\mathcal{L}_n \otimes \mathcal{L}_n \to \mathcal{R}_n$ we described earlier is a homomorphism between representation bundles over X_n which corresponds to the Lie algebra fact that $\mathbf{R}_n$ is an irreducible component of $\mathbf{L}_n \otimes \mathbf{L}_n$.

2.4 Decomposing $\mathcal{L}_n$ under $\pi^* L\mathcal{E}_{n-1}$

We fix a line L on X_n and we denote the blow down morphism of X_n along L as $\pi : X_n \to X_{n-1}$. We also denote the image of L as $p \in X_{n-1}$. In the previous subsection we decompose $L\mathcal{E}_n$ under the Lie subalgebra bundle $\pi^* L\mathcal{E}_{n-1}$. Now we discuss a similar decomposition for $\mathcal{L}_n$. Notice that choosing a fix line on X_n break the symmetry among all lines on X_n and this decomposition of $\mathcal{L}_n$ reflects the intersection properties of L with other lines on X_n.

First we discuss the decomposition of the representation $\mathbf{L}_n$ under the Lie subalgebra E_{n-1}, the branching rule. As a representation of E_{n-1}, we have $\mathbf{L}_n = \mathbf{L}_{n-1} + \mathbf{R}_{n-1} + \mathbf{1}$ when $n \leq 6$. Here $\mathbf{1}$ denote the trivial representation of dimension one. We also have $\mathbf{L}_7 = \mathbf{L}_6 + \mathbf{R}_6 + \mathbf{1} + \mathbf{1}$. When $n = 8$ then $E_7 + A_1$ is a maximal Lie subalgebra of E_8 and as a representation of this subalgebra we have the decomposition $\mathbf{L}_8 = \mathbf{L}_7 \otimes \Lambda_1 + \mathbf{R}_7 + A_1$ where Λ_1 is the standard representation of $A_1 = \mathbf{sl}(2)$.

First we notice that $\pi^* \mathcal{L}_{n-1}$ is a subbundle of $\mathcal{L}_n$. This is equivalent to the statement that p does not lie on any line on X_{n-1}. Otherwise the strict

transform of the line containing p would be a (-2)-curve and the existence of such curve violates the ampleness assumption of the anti-canonical class of X_n. Second, it is obvious that $O(L)$ is also a subbundle of $\mathcal{L}_n$.

Third if R determines a ruling on X_{n-1}, then there is a unique fiber of the ruling which contains the point p. By similar reason as above, this must be a smooth fiber of the ruling. The strict transform of this smooth fiber containing p is a line on X_n. Hence there is a line in the divisor class $\pi^* R - L$. Now we recall that the representation bundle of $L\mathcal{E}_{n-1}$ corresponding to $\mathbf{R}_{n-1}$ is given by

$$\begin{aligned} \mathcal{R}_{n-1} &= \bigoplus_{R:\text{ruling}} O_{X_{n-1}}(R) \text{ when } n \leq 7 \\ \mathcal{R}_7 &= \bigoplus_{R:\text{ruling}} O_{X_7}(R) + O(-K)^{\oplus 7}. \end{aligned}$$

Therefore, when $n \leq 7$, then $\pi^*\mathcal{L}_{n-1} + \pi^*\mathcal{R}_{n-1} \otimes O(-L) + O(L)$is a subbundle of $\mathcal{L}_n$. In fact if $n \leq 6$ this exhaust the whole bundle $\mathcal{L}_n$. This is because $l \cdot L = -1, 0$ or 1 for any line l on X_n. When $l \cdot L = -1$ (resp. 0 and 1), then l is L (resp. constructed using $\mathcal{L}_{n-1}$ and $\mathcal{R}_{n-1}$). When $n = 7$, there is a unique line l on X_7 such that $l \cdot L = 2$. Namely l and L forms a 2-gon. That is, l is linearly equivalent to $-K - L$. When $n = 8$, the situation is slightly more involved and we will discuss it later. In conclusion we have

$$\begin{array}{lll} \mathcal{L}_n = \pi^*\mathcal{L}_{n-1} & +\pi^*\mathcal{R}_{n-1} \otimes O(-L) + O(L) & \text{when } n \leq 6, \\ \mathcal{L}_7 = \pi^*\mathcal{L}_6 & +\pi^*\mathcal{R}_6 \otimes O(-L) + O(L) + O(-K-L), & \\ \mathcal{L}_8 = \pi^*\mathcal{L}_7 \otimes \Lambda_1^* & +\pi^*\mathcal{R}_7 \otimes O(-L) + \mathcal{A}_1 \otimes O(-K). & \end{array}$$

Moreover these decompositions of vector bundles are in fact decompositions of representation bundles of $L\mathcal{E}_{n-1}$.

For example the ten lines on a cubic surface X_6 intersecting L forms five pairs of intersecting lines (see e.g. [Hartshorne] and [Reid2]). It is reflecting the fact that the representation $\mathbf{R}_5$ of E_5 is the standard representation of $\mathbf{so}(10)$ under the identification of E_5 with $D_5 = \mathbf{so}(10)$ and therefore it carries a natural quadratic form. See section six for details. Moreover each line on the cubic surface determines a unique ruling on it and vice versa. More precisely we have $\mathcal{R}_6 = \mathcal{L}_6^* \otimes O(-K_{X_6})$. This isomorphism corresponds to the isomorphism $\mathbf{R}_6 \cong \mathbf{L}_6^*$ via the outer-automorphism of the Dynkin diagram of E_6. Then we can rewrite the above decomposition of $\mathcal{L}_7$ as

$$\mathcal{L}_7 = (\pi^*\mathcal{L}_6 + O(-L)) \otimes (O + O(-K)).$$

To see this geometrically, we consider a projection of the cubic surface X_6 away from a point q on it. Then the projection defines a double cover of $\mathbb{P}^2$ branched along a quartic plane curve C. The 27 lines on X_6 together with L on

X_7 project to the 28 bitangents to the quartic plane curve. The pre-image of each bitangent to C is a 2-gon on X_7. These informations are neatly packaged in the above decomposition of $\mathcal{L}_7$.

Next we want to verify that the Lie algebra bundle associated to each of the E_n-bundle $\mathcal{E}_n$ on X_n is precisely $L\mathcal{E}_n$. This is because each of those $\mathcal{E}_n$'s are defined as automorphism bundles of certain fiberwise algebraic structure on representation bundles of $L\mathcal{E}_n$. Moreover such algebraic structures are also preserved by $L\mathcal{E}_n$ and therefore the identification of $Lie\,(\mathcal{E}_n)$ and $L\mathcal{E}_n$ follows from corresponding statements in representation theory of Lie algebra (see e.g. [Adams]).

2.5 Small n examples

When $n = 1$ then X_1 is just the blowup of $\mathbb{P}^2$ at one point and there is a unique line l on X_1, namely the exceptional curve for the blowup morphism. Also X_1 has a unique ruling determined by $H - l$ where H is the pullback of the hyperplane class from $\mathbb{P}^2$. In this case we have $L\mathcal{E}_1 = O$, $\mathcal{L}_1 = O\,(l)$ and $\mathcal{R}_1 = O\,(H - l)$.

When $n = 2$ there are 3 lines on X_2. We denote them as l_1, l_2 and l_3. They are divided into two different types. By rearranging indexes, we can assume that $l_1 \cdot l_3 = l_2 \cdot l_3 = 1$ and $l_1 \cdot l_2 = 0$. Notice that this is the only time that lines on X_n are divided into different types. This can translated into reducibility of $\mathcal{L}_2$ as a representation bundle of $L\mathcal{E}_2$.

Recall that $E_2 = A_1 + \mathbf{u}\,(1)$ and we can write $L\mathcal{E}_2 = L\mathcal{A}_1 + O$ where $L\mathcal{A}_1 = O + O\,(l_1 - l_2) + O\,(l_2 - l_1)$ is the Lie algebra bundle of type A_1. From definitions we have $\mathcal{L}_2 = O\,(l_1) + O\,(l_2) + O\,(l_3)$ and $\mathcal{R}_2 = O\,(l_1 + l_3) + O\,(l_2 + l_3)$. As representations of A_1 we have $\mathbf{L}_2 = \mathbf{R}_2 + \mathbf{1}$. Similarly, as representation bundles of $L\mathcal{A}_1$ we have $\mathcal{L}_2 = \mathcal{R}_2 \otimes O\,(-l_3) + O\,(l_3)$.

If we blow down X_2 along l_1 (or l_2), then we get X_1 and the E_1-bundle $L\mathcal{E}_1$ over it with $E_1 = \mathbf{u}\,(1)$ as before. However we can also blow down X_2 along l_3, then we get the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Now we should interpret E_1 as the Lie subalgebra A_1 of E_2 instead of $\mathbf{u}\,(1)$. To prevent confusion, we call it $\overline{E}_1$ and we also write $\overline{\mathbf{L}}_1$, $\overline{\mathbf{R}}_1$ for its representations. In fact $\overline{\mathbf{L}}_1$ is trivial and $\overline{\mathbf{R}}_1$ is the standard representation of it under the identification of $\overline{E}_1$ with $A_1 = \mathbf{sl}\,(2)$.

As for $\mathbb{P}^1 \times \mathbb{P}^1$, it has no lines and two rulings. Therefore $\overline{\mathcal{L}}_1$ is trivial and $\overline{\mathcal{R}}_1$ is a rank two bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover $L\overline{\mathcal{E}}_1 = End_0\left(\overline{\mathcal{R}}_1\right)$. The previous decomposition of $L\mathcal{E}_n$ and $\mathcal{L}_n$ under $L\mathcal{E}_{n-1}$ also hold true in this case. Since $\overline{\mathcal{L}}_1$ is trivial we have

$$\begin{aligned} L\mathcal{E}_2 &= \pi^* L\overline{\mathcal{E}}_1 + O, \\ \mathcal{L}_2 &= \pi^* \overline{\mathcal{R}}_1 \otimes O\,(-L) + O\,(L)\,. \end{aligned}$$

Here $L = l_3$. We can also write down the decomposition for $\mathcal{R}_2$, namely $\mathcal{R}_2 = \pi^* \overline{\mathcal{R}}_1$.

When $n = 3$ the non-simpleness of E_3 is related to Cremona transformations on X_3. First we represent X_3 as a blowup of $\mathbb{P}^2$ at three generic points, $\pi : X_3 \to \mathbb{P}^2$ and denote the exceptional locus as $L_1 \cup L_2 \cup L_3$. Notice that the Lie algebra E_3 is not simple: $E_3 = \mathbf{sl}(3) + \mathbf{sl}(2)$ and $\mathbf{L}_3$ is the tensor product of standard representation of $\mathbf{sl}(3)$ and $\mathbf{sl}(2)$. Geometrically we have $L\mathcal{E}_3 = \mathcal{A}_3 + \mathcal{A}_2$ where $\mathcal{A}_3 = 2O \oplus_{i \neq j} O(L_i - L_j)$ is a $\mathbf{sl}(3)$ Lie algebra bundle over X_3 and $\mathcal{A}_2 = O + O(H - L_1 - L_2 - L_3) + O(-H + L_1 + L_2 + L_3)$ is a $\mathbf{sl}(2)$ Lie algebra bundle over X_3. For $\mathcal{L}_3$ we also have $\mathcal{L}_3 = \mathcal{W}_3 \otimes \mathcal{W}_2$ with $\mathcal{W}_3 = O(L_1) + O(L_2) + O(L_3)$ and $\mathcal{W}_2 = O + O(H - L_1 - L_2 - L_3)$ being representation bundles of $\mathcal{A}_3$ and $\mathcal{A}_2$ respectively.

We can also write $\mathcal{L}_3 = \mathcal{W}_3' \otimes \mathcal{W}_2^*$ with $\mathcal{W}_3' = O(H - L_2 - L_3) + O(H - L_3 - L_1) + O(H - L_1 - L_2)$. The symmetric relation between $\mathcal{W}_3$ and $\mathcal{W}_3'$ is related to the Cremona transformation.

When $n = 4$ the ranks of $\mathcal{L}_4$ and $\mathcal{R}_4$ equals ten and five respectively. Notice that $E_4 = A_4 = \mathbf{sl}(5)$ and the representation $\mathbf{R}_4$ of E_4 corresponds to the standard representation of $\mathbf{sl}(5)$. Therefore we expect that $L\mathcal{E}_4 = End_0(\mathcal{R}_4)$, the bundle of trace-free endomorphisms of $\mathcal{R}_4$. To see this directly, we represent X_4 as a blowup of $\mathbb{P}^4$ at four points and we denote the exceptional locus of this blowup as $L_1 \cup L_2 \cup L_3 \cup L_4$, then we have

$$\mathcal{L}_4 = \oplus_{i=1}^4 O(L_i) + \oplus_{1 \leq i < j \leq 4} O(H - L_i - L_j)$$

and

$$\mathcal{R}_4 = \oplus_{i=1}^4 O(H - L_i) + O(2H - L_1 - L_2 - L_3 - L_4).$$

By direct computations, we have

$$\begin{aligned} End_0(\mathcal{R}_4) &= O_X^{\oplus 4} + \oplus_{i \neq j}(L_i - L_j) + \oplus_{i=1}^4 \left(O\left(H - \sum_{j=1}^4 L_j + L_i\right) + O\left(-H + \sum_{j=1}^4 L_j - L_i\right)\right) \\ &= O_X^{\oplus 4} + \bigoplus_{\substack{D = l - l' \\ l, l' \text{ disjoint lines on } X_4}} O(D) \\ &= O_X^{\oplus 4} + \bigoplus_{\substack{D^2 = -2 \\ D \cdot K = 0}} O(D). \end{aligned}$$

We also have the isomorphism

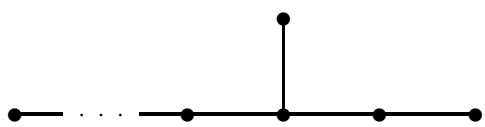
$$\Lambda^3 \mathcal{R}_4 = \mathcal{L}_4 \otimes O(-K).$$

This corresponds to the fact that the fundamental representation $\mathbf{L}_4$ of $\mathbf{sl}(5)$ equals the third exterior power of its standard representation.

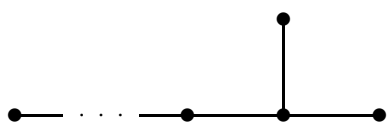
3 D_{n-1}-bundle over ruled surfaces

3.1 Ruling on X_n and construction of $\mathcal{D}_{n-1}$

The E_n-bundle $\mathcal{E}_n$ over X_n would have its structure group reduces to a smaller subgroup of E_n if X_n admits additional geometric structure. In this section, we study the subalgebra $\mathbf{so}\,(2n-2) = D_{n-1} \subset E_n$. This corresponds to removing the node on the right end in the Dynkin diagram of E_n.



$: E_n$



$: D_{n-1}$

Recall that this right end node corresponds to the fundamental representation $\mathbf{R}_n$ of E_n. Its associated representation bundle $\mathcal{R}_n$ is constructed using the set of rulings on X_n. Now we fix one ruling on X_n. That is we pick a divisor R on X_n which satisfies $R^2 = 0$ and $R \cdot K_{X_n} = -2$. We shall see that such a geometric structure reduces the structure group of $\mathcal{E}_n$ from E_n to D_{n-1}. We remark that the construction of the D_{n-1}-bundle over X_n applies to blowups of $\mathbb{P}^2$ at arbitrary number of points.

Let $X = X_n$ be a blowup of $\mathbb{P}^2$ at n generic points and let R be a divisor on X_n defining a ruling .We define a vector bundle on X_n as follow:

$$\mathcal{W}_{n-1} = \bigoplus_{\substack{C^2=-1 \\ CK=-1 \\ CR=0}} O_X\,(C)\,.$$

Then $\mathcal{W}_{n-1}$ is a subbundle of $\mathcal{L}_n$ for $n \leq 8$. In general the rank of $\mathcal{W}_{n-1}$ equals $2n-2$ and it carries a natural fiberwise non-degenerate quadratic form $q : \mathcal{W}_{n-1} \otimes \mathcal{W}_{n-1} \to O_{X_n}\,(R)$. We denote the connected component of the automorphism bundle of $\mathcal{W}_{n-1}$ preserving q as $\mathcal{D}_{n-1}$.

To see this we need to look at the ruling $\pi : X_n \to \mathbb{P}^1$ determined by R. Namely the fiber divisor class of π equals R. Genericity of X_n implies that each singular fiber of π consists of two irreducible components. Each of them is a (-1) curve and they meet at a single point transversely.

Since the Euler number of X_n equals $n+3$, the number of singular fibers is $n-1$. We denote them as $L_1 \cup L_1', L_2 \cup L_2', ..., L_{n-1} \cup L_{n-1}'$. For any $C = L_i$ or L_i', it satisfies $C^2 = -1, C \cdot K_X = -1$ and $C \cdot R = 0$. It is not difficult to show that the converse is also true. Hence $\mathcal{W}_{n-1} = \oplus_{i=1}^{n-1} (O_{X_n}(L_i) + O_{X_n}(L_i'))$. Using the isomorphisms $O_{X_n}(L_i) \otimes O_{X_n}(L_i') \cong O_{X_n}(R)$, we obtain a non-degenerated fiberwise quadratic form

$$q_{n-1} : \mathcal{W}_{n-1} \otimes \mathcal{W}_{n-1} \to O_{X_n}(R).$$

In terms of the above decomposition of $\mathcal{W}_{n-1}$, we have $q_{n-1} = \oplus^{n-1} \mathbf{H}$ with $\mathbf{H} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

It is easy to see that the adjoint bundle of $\mathcal{D}_{n-1}$ is given by

$$L\mathcal{D}_{n-1} = \Lambda^2 \mathcal{W}_{n-1} \otimes O_{X_n}(-R).$$

Using the identities $O(L_i + L_j - R) \cong O(L_i - L_j')$ and $O(L_i + L_j' - R) \cong O(L_i - L_j)$, we obtain

$$L\mathcal{D}_{n-1} = O_X^{\oplus n-1} \bigoplus_{\substack{C^2=-2 \\ CK=0 \\ CR=0}} O_X(C).^2$$

From this description, it is obvious that $L\mathcal{D}_{n-1}$ is a vector subbundle of $L\mathcal{E}_n$ when $n \leq 8$. In fact $L\mathcal{D}_{n-1}$ is a Lie subalgebra bundle of $L\mathcal{E}_n$.

3.2 Spinor bundles of $\mathcal{D}_{n-1}$

Using $\Lambda^l \mathcal{W}_{n-1}$ with $l = 1, 2, ..., n-3$, we can construct associated bundles of $\mathcal{D}_{n-1}$ corresponding to every fundamental representation of D_{n-1} except the two spinor representations S^+ and S^-. In fact, these two spinor bundles $\mathcal{S}^+$ and $\mathcal{S}^-$ over X_n will be needed later when we decompose $L\mathcal{E}_n$ and $\mathcal{L}_n$ under restriction to $L\mathcal{D}_{n-1}$. We leave them as exercise for readers to check directly that

$$\mathcal{S}^+ = \bigoplus_{\substack{S^2=-1 \\ SK=-1 \\ SR=1}} O_{X_n}(S) \text{ and } \mathcal{S}^- = \bigoplus_{\substack{T^2=-2 \\ TK=0 \\ TR=1}} O_{X_n}(T).$$

In particular divisor S as above correspond to line on X which is a section of the ruling R. Similar for any divisor T as above, the divisor $T+R$ corresponds to a ruling on X which is a section of the given ruling R. In fact $n \leq 5$, then every line on X_n is either a section of the ruling or is a component of a singular fiber of the ruling. This gives rise to the decomposition $\mathcal{L}_n = \mathcal{W}_{n-1} + \mathcal{S}^+$, for $n \leq 5$. On the other hand, adding to R a divisor T as above gives us another ruling of X_n. When $n \leq 4$, then determines all rulings on X_n. This gives rise to the decomposition $\mathcal{R}_n = O(R) \otimes (O + \mathcal{S}^-)$, for $n \leq 4$.

Example 16 [D_3-*bundle over* X_4] *case: Let* $X = X_4$ *be the blowup of* $\mathbb{P}^2$ *at four generic points. Recall that, under Aut* $\left(\mathbb{P}^2\right) = \mathbb{P}GL\left(3, \mathbb{C}\right)$*, any four generic points on* $\mathbb{P}^2$ *is equivalent to any others. Let us denote the exceptional locus of this blowup as* L_0, L_1, L_2 *and* L_3*. We also denote the pullback of the hyperplane class from* $\mathbb{P}^2$ *as* H*. Then* $R = H - L_0$ *satisfies* $R^2 = 0$ *and* $R \cdot K_X = -2$ *and hence defines a ruling on* X*.*

In this ruling, there are three singular fibers $L_1 \cup L_1', L_2 \cup L_2'$ *and* $L_3 \cup L_3'$*. The linear equivalent classes of* L_i' *are given by* $L_i' = H - L_0 - L_i$ *for* $i = 1, 2, 3$*. Moreover* L_0 *is a section of the ruling.*

Bundle over X_4 :	$\mathcal{W}_3$	$\mathcal{S}^+$	$\mathcal{S}^-$
divisors for its direct summands	L_1 L_2 L_3 $H - L_0 - L_1$ $H - L_0 - L_2$ $H - L_0 - L_3$	L_0 $H - L_1 - L_2$ $H - L_2 - L_3$ $H - L_3 - L_1$	$L_0 - L_1$ $L_0 - L_2$ $L_0 - L_3$ $H - L_1 - L_2 - L_3$.

Example 17 [D_4-*bundle over* X_5] *case: We use notations similar to the previous example and we have the following table:*

Bundle over X_5 :	$\mathcal{W}_4$	$\mathcal{S}^+$	$\mathcal{S}^-$
divisors for its direct summands	L_1 L_2 L_3 L_4 $H - L_0 - L_1$ $H - L_0 - L_2$ $H - L_0 - L_3$ $H - L_0 - L_4$	L_0 $H - L_1 - L_2$ $H - L_1 - L_3$ $H - L_1 - L_4$ $H - L_2 - L_3$ $H - L_2 - L_4$ $H - L_3 - L_4$ $2H - \sum_{i=0}^{4} L_i$	$L_0 - L_1$ $L_0 - L_2$ $L_0 - L_3$ $L_0 - L_4$ $H - L_2 - L_3 - L_4$ $H - L_1 - L_3 - L_4$ $H - L_1 - L_2 - L_4$ $H - L_1 - L_2 - L_3$.

In both these examples, we see that the union of the first two columns exhausts all lines on X_4 and X_5. As we will explain, the situations are more complicated for X_6, X_7 and X_8.

Clifford multiplication

In linear algebra the product of a vector and a spinor produces a spinor with a different parity. For our bundles of spinors over X_n, we have homomorphisms

$$\mathcal{S}^+ \otimes \mathcal{W}_{n-1}^* \to \mathcal{S}^-,$$
$$\mathcal{S}^- \otimes \mathcal{W}_{n-1} \to \mathcal{S}^+,$$

corresponding to fiberwise Clifford multiplications[3]. It is very easy to describe these homomorphisms explicitly. Let $O_X\left(S\right)$ (and $O_X\left(-C\right)$) be a direct summand of $\mathcal{S}^+$ (and $\mathcal{W}_{n-1}^*$). If $S \cdot C = 0$ then $T = S - C$ satisfies

[3] $\mathcal{W}_{n-1} = \mathcal{W}_{n-1}^* \otimes O_{X_n}\left(R\right)$.

$T^2 = -2, T \cdot K_X = 0$ and $T \cdot R = 1$. That is $O_X(S) \otimes O_X(C)^* = O_X(T) \subset \mathcal{S}^-$. We set the product of elements from other types of direct summands to be zero. This gives the first Clifford multiplication homomorphism. The second homomorphism can be described in the same way.

Geometrically these homomorphisms reflect the fact that an irreducible component of a singular fiber of the given ruling together with a line section passing through it determines another ruling which is a section of the given ruling. Moreover all ruling sections arise in this manner.

Dualities

When $n = 2m$ is even, we have an isomorphism

$$\left(\mathcal{S}^+\right)^* \otimes O_{X_{2m}}\left((m-4)R - K_X\right) \cong \mathcal{S}^-.$$

This follows from earlier explicit descriptions of $\mathcal{S}^+$ and $\mathcal{S}^-$. This isomorphism corresponds to the linear algebra fact that the two spinor representations of $Spin\,(4m-2)$ are dual to each other.

Similarly when $n = 2m+1$ is odd, we have isomorphisms

$$\left(\mathcal{S}^+\right)^* \otimes O_{X_{2m+1}}\left((m-3)R - K_X\right) \cong \mathcal{S}^+,$$

and

$$\left(\mathcal{S}^-\right)^* \otimes O_{X_{2m+1}}\left((m-4)R - K_X\right) \cong \mathcal{S}^-.$$

These isomorphisms correspond to linear algebra facts that the two spinor representations of $Spin\,(4m)$ are self-dual. If we rewrite these two homomorphisms as

$$\mathcal{S}^+ \otimes \mathcal{S}^+ \to O_{X_{2m+1}}\left((m-3)R - K_X\right),$$

and

$$\mathcal{S}^- \otimes \mathcal{S}^- \to O_{X_{2m+1}}\left((m-4)R - K_X\right).$$

We obtain fiberwise non-degenerate quadratic forms on $\mathcal{S}^+$ and $\mathcal{S}^-$. On the one hand, it is related to the fact that spinor representations $\mathcal{S}^\pm$ of $Spin\,(4m)$ are of real type. On the other hand, they give inner product structure on $\mathcal{S}^+$ and $\mathcal{S}^-$ which will become important later. There are many other algebraic structures on spinor representations of D_{n-1} which can be translated to our bundle version in a similar manner. Instead of writing out each of them, we are going to look at relationship between $\mathcal{E}_n$ and $\mathcal{D}_{n-1}$. Such relationship is very interesting because they reflect the configuration of lines on X_n in relation to a fixed ruling on X_n.

3.3 Reduction from E_n to D_{n-1}

Decompose $\mathcal{L}_n$ under $\mathcal{D}_{n-1}$

Recall that the bundle $\mathcal{L}_n$ over X_n is constructed from the collection of lines on X_n. When R determines a ruling on X_n, every line would have non-negative intersection with R. However this intersection number cannot be too big.

Proposition 18 *If L is a line on X_n with a ruling determined by R, then $L \cdot R \leq 1$ if $n \leq 5$*

Proof of proposition: When $n \leq 6$ any two lines of X_n intersect at not more than one point. Now, from the ampleness assumption of K^{-1}, every singular fiber of the ruling consists of two lines. Therefore $L \cdot R$ is at most two. However if $L \cdot R = 2$ then the union of L with any singular fiber of the ruling will form a triangle. But we know from proposition 11 that triangle does not exist when $n \leq 5$. Hence $L \cdot R \leq 1$. $\square$

Remark 19 *From the proof of the proposition, we can also see that in the case of X_6 there is only one L with $L \cdot R = 2$, otherwise $L \cdot R \leq 1$. It is because the union of L with any singular fiber of the ruling forms a triangle which is necessary an anti-canonical divisor. Hence $L = -K - R$.*

In fact, with more care, we can show that $L \cdot R \leq 2$ even when $n = 7$.

From the proposition, we have $L \cdot R = 0$ or 1 for any line L on X_n with $n \leq 5$. Line bundles associated to these divisors are precisely direct summands of $\mathcal{W}_{n-1}$ and $\mathcal{S}^+$ respectively. Therefore we have isomorphism

$$\mathcal{L}_n = \mathcal{W}_{n-1} + \mathcal{S}^+ \text{ when } n \leq 5.$$

In fact these are isomorphisms as representation bundles of $\mathcal{D}_{n-1}$. Of course, these isomorphisms corresponds to the decomposition into irreducible summands of $\mathbf{L}_n$ when we restrict the Lie algebra from E_n to D_n :

$$\mathbf{L}_n|_{D_{n-1}} = W_{n-1} + S^+$$

for $n \leq 5$.

Example 20 (D_3-bundle over X_4) *In this case, after choosing a ruling R, we have $\mathcal{L}_4 = \mathcal{W}_3 + \mathcal{S}^+$. The four lines corresponding to $\mathcal{S}^+$ are sections of the ruling. We call them L_1, L_2, L_3 and L_4. They are disjoint lines on X_4. It is because every line on X_4 intersects three other lines (from section two that $\mathcal{L}_4 = \mathcal{L}_3 + \mathcal{R}_3 + L$ and $\dim \mathbf{R}_3 = 3$) and when it is a section, it intersects the three singular fibers of the ruling. Each singular fiber consists of two lines and therefore all sections are disjoint. We can blowdown these L_i's and we obtain $\mathbb{P}^2$. Then the ruling R equals $2H - L_1 - L_2 - L_3 - L_4$ where $H = -K - R$ is the pullback of the hyperplane class on $\mathbb{P}^2$. If we denote L_{ij} the line on X_4 in the class $H - L_i - L_j$, then the three singular fibers of the rulings are*

$L_{12} \cup L_{34}, L_{13} \cup L_{42}$ and $L_{14} \cup L_{23}$. Moreover L_{ij} meets L_k if and only if $k = i$ or j.

In terms of $L\mathcal{D}_3$ representation bundles over X_4, we obtain $\mathcal{W}_3 = \Lambda^2 \mathcal{S}^+ \otimes O(K + 2R)$ and $\mathcal{L}_4 = \mathcal{S}^+ + \Lambda^2 \mathcal{S}^+ \otimes O(K + 2R)$. This reflects the coincidence $D_3 = A_3$ or $\mathbf{so}(6) = \mathbf{sl}(4)$. It is because S^+ is the standard representation of $\mathbf{sl}(4)$ and therefore $W_3 = \Lambda^2 S^+$. Similarly we have $\mathcal{S}^- = \Lambda^3 \mathcal{S}^+ \otimes O(K + R)$ corresponding to $S^- = \Lambda^3 S^+$ as representation of D_3.

We now come back to discuss the decomposition of $\mathbf{L}_n$ when $n = 6$, we have

$$\mathbf{L}_6|_{D_5} = \mathbf{1} + W_5 + S^+.$$

On the representation bundle level, we have

$$\mathcal{L}_6 = O_{X_6}(-K_X - R) + \mathcal{W}_5 + \mathcal{S}^+.$$

From the proof of the previous proposition, there is a unique line L on X_6 with $L \cdot R > 1$. In fact we have $L \cdot R = 2$ and its divisor class is linearly equivalent to $-K_{X_6} - R$. The rank decomposition of $\mathcal{L}_6$ is quite familiar: $27 = 1 + 10 + 16$. This reminds us the fact that, given any line L on a cubic surface, there are exactly 10 lines intersecting L and 16 lines disjoint from L. This translate into $\mathcal{L}_6 = O_{X_6}(L) + \pi^* \mathcal{R}_5 \otimes O(-L) + \pi^* \mathcal{L}_5$, where $\pi : X_6 \to X_5$ is the blow down of L.

To explain it, we remind ourselves that there is an outer-automorphism of E_6 which corresponds to the $\mathbb{Z}_2$ symmetry of the Dynkin diagram of E_6. In particular it interchanges the two nodes corresponding to fundamental representations $\mathcal{R}_6$ and $\mathcal{L}_6$. In terms of the geometry of the cubic surface X_6, it establishes a correspondence between rulings on X_6 and lines on X_6. Concretely, given any divisor R on X_6 which determines a ruling, there is a unique line L with $L \cdot R = 2$ as we mentioned earlier. Moreover the two decompositions of $\mathcal{L}_6$ are identified under this $\mathbb{Z}_2$ symmetry. Namely $\mathcal{W}_5 \cong \pi^* \mathcal{R}_5 \otimes O(-L)$ and $\mathcal{S}^+ \cong \pi^* \mathcal{L}_5$.

When $n = 7$ or 8, the decompositions for the representations $\mathbf{L}_n$ are different. Namely

$$\mathbf{L}_7|_{D_6 \times A_1} = W_6 \otimes \Lambda_1 + S^+$$

and

$$\mathbf{L}_8|_{D_8} = W_8 + S^+.$$

The corresponding decompositions of representation bundles are

$$\mathcal{L}_7 = \mathcal{W}_6 \otimes \Lambda_1 + \mathcal{S}^+,$$

and

$$\mathcal{L}_8 = \mathcal{W}_8 + \mathcal{S}^+.$$

The details for these two cases will be discussed in later sections.

Decompose $\mathcal{R}_n$ under $\mathcal{D}_{n-1}$

We have the following decomposition of $\mathcal{R}_n$ as a representation bundle of $L\mathcal{D}_{n-1}$:

$$\begin{aligned}
\mathcal{R}_n &= O(R)\left(O+\mathcal{S}^-\right), \quad \text{for } n \leq 4, \\
\mathcal{R}_5 &= O(R)\left(O+\mathcal{S}^- + O(-K-2R)\right), \\
\mathcal{R}_6 &= O(R)\left(O+\mathcal{S}^- + \mathcal{W}_5 \otimes O(-K-2R)\right), \\
\mathcal{R}_7 &= O(R)\left(S^2\Lambda_1 + \mathcal{S}^- \otimes \Lambda_1 + \Lambda^2\mathcal{W}_6 \otimes (-K-2R)\right).
\end{aligned}$$

For $\mathcal{R}_7$ we decompose it under $L\mathcal{D}_6 + L\mathcal{A}_1$. If we restrict to any fiber, it recovers the Lie algebra facts:

$$\begin{aligned}
\mathbf{R}_n &= \mathbf{1} + S^-, \quad \text{for } n \leq 4 \\
\mathbf{R}_5 &= \mathbf{1} + S^- + \mathbf{1}, \\
\mathbf{R}_6 &= \mathbf{1} + S^- + W_5, \\
\mathbf{R}_7 &= \mathbf{1} \otimes S^2\Lambda_1 + S^- \otimes \Lambda_1 + \Lambda^2 W_6 \otimes \mathbf{1}.
\end{aligned}$$

We see from the above decompositions that, after we fix a ruling R on X_n, then other rulings on X_n can be obtained by adding an element of $\mathcal{S}^-$. In fact, when n is not bigger than four, it gives all rulings on X_n. First if $O(T)$ is a direct summand of $\mathcal{S}^-$, namely T satisfies $T^2 = -2, T \cdot K = 0$ and $T \cdot R = 1$, then it is easy to check that $R' = R + T$ satisfies $R'^2 = 0, R' \cdot K = -2$. Hence R' is another ruling on X_n. Moreover $R \cdot R' = 1$, that is R' is a section of the ruling determined by R. To prove the reverse direction, we need the following geometric proposition.

Proposition 21 *If we fix a ruling R on X_n, then any other ruling R' is a section of it, i.e. $R \cdot R' = 1$, provided that $n \leq 4$.*

When $n = 5$, then either R' is a section or $R' = -K - R$. In the latter case, R' is a 2-section, i.e. $R \cdot R' = 2$.

Proof of proposition: We assume that R and R' are two rulings on X_n. Without loss of generality we assume that both R and R' are singular fibers of the corresponding ruling. In particular each of them consists of two lines. Recall that,when $n \leq 6$, every two lines intersect at not more than one point. Therefore if $R \cdot R' > 1$, either $R \cup R'$ is a rectangle or it contains a triangle subconfiguration. There is no triangle when $n \leq 5$ and there is no rectangle when $n \leq 4$ and the only type of rectangle on X_5 is an anti-canonical divisor. This proves the proposition. □

Now if R' is another ruling with $R \cdot R' = 1$ then, by reversing previous arguments, $O(T)$ is a direct summand of $\mathcal{S}^-$. Therefore we have verified above decompositions of $\mathcal{R}_n$ for $n \leq 5$. For $n = 6$, we have isomorphism $\mathcal{R}_6 = \mathcal{L}_6^* \otimes O(-K)$ and hence the decomposition for $\mathcal{L}_6$ under $L\mathcal{D}_5$ gives the one for $\mathcal{R}_6$. For $n = 7$, there is an isomorphism $\mathcal{R}_7 = L\mathcal{E}_7 \otimes O(-K)$. In section seven,

we will discuss a decomposition of $L\mathcal{E}_7$ under $L\mathcal{D}_6 + L\mathcal{A}_1$ and hence we obtain a decomposition for $\mathcal{R}_7$ which is the one given above.

Decompose $L\mathcal{E}_n$ under $L\mathcal{D}_{n-1}$

We have the following decomposition of $L\mathcal{E}_n$ as a representation bundle of $L\mathcal{D}_{n-1}$ (or $L(\mathcal{D}_7 + \mathcal{A}_1)$, or $L\mathcal{D}_8$):

$$\begin{aligned} L\mathcal{E}_n &= L\mathcal{D}_{n-1} + O + \mathcal{S}^- + \left(\mathcal{S}^-\right)^*, \quad \text{for } n \leq 6, \\ L\mathcal{E}_7 &= L\mathcal{D}_7 + L\mathcal{A}_1 + \mathcal{S}^- \otimes \Lambda_1^*, \\ L\mathcal{E}_8 &= L\mathcal{D}_8 + \mathcal{S}^+. \end{aligned}$$

Notice that when $n = 2m$ is even, we have an isomorphism $(\mathcal{S}^-)^* = \mathcal{S}^+ \otimes O((4-m)R + K)$ and when $n = 2m+1$ is odd, we have an isomorphism $(\mathcal{S}^-)^* = \mathcal{S}^- \otimes O((4-m)R + K)$. Moreover in the case of X_7, we have an isomorphism $\Lambda_1^* = \Lambda_1 \otimes O(R + K)$ (see section seven). So we can express the decompositions of $L\mathcal{E}_n$ in terms of representation bundles of $L\mathcal{D}_{n-1}$ using only fundamental representations of D_{n-1}. More precisely, we have

$$\begin{aligned} L\mathcal{E}_n &= L\mathcal{D}_{n-1} + O + \mathcal{S}^- + \mathcal{S}^- \otimes O((4-m)R + K), \quad \text{for } n = 2m+1 \leq 5, \\ L\mathcal{E}_n &= L\mathcal{D}_{n-1} + O + \mathcal{S}^- + \mathcal{S}^+ \otimes O((4-m)R + K), \quad \text{for } n = 2m \leq 6, \\ L\mathcal{E}_7 &= L\mathcal{D}_7 + L\mathcal{A}_1 + \mathcal{S}^- \otimes \Lambda_1 \otimes O(R + K), \\ L\mathcal{E}_8 &= L\mathcal{D}_8 + \mathcal{S}^+. \end{aligned}$$

If we restrict to any fiber, it recovers the Lie algebra facts:

$$\begin{aligned} E_n &= D_{n-1} + \mathbf{1} + S^- + S^-, \quad \text{for } n = 2m+1 \leq 5, \\ E_n &= D_{n-1} + \mathbf{1} + S^- + S^+, \quad \text{for } n = 2m \leq 6, \\ E_7 &= D_7 + LA_1 + S^- \otimes \Lambda_1, \\ E_8 &= D_8 + S^+. \end{aligned}$$

Now to show that $L\mathcal{E}_n = L\mathcal{D}_{n-1} + O + \mathcal{S}^- + (\mathcal{S}^-)^*$, for $n \leq 6$, we need the following geometric proposition.

Proposition 22 *If we fix a ruling R on X_n, then any D with $D^2 = -2$ and $D \cdot K = 0$ satisfies $|D \cdot R| \leq 1$ provided that $n \leq 6$.*

Proof of proposition: We can assume that $D = l_1 - l_2$ with $l_1 \cdot l_2 = 0$. By considering a singular fiber of the ruling, we can assume that $R = l_3 + l_4$ with $l_3 \cdot l_4 = 1$. Here l_i's are all lines on X_n. We suppose that $D \cdot R = (l_1 - l_2) \cdot (l_3 + l_4) \geq 2$. We first assume that all l_i's are distinct. Then $(l_1 - l_2) \cdot (l_3 + l_4) \leq l_1 \cdot (l_3 + l_4) \leq 2$ because two lines on X_n intersect at not more than one point when $n \leq 6$. Therefore all inequality sign are equality and we get $l_2 \cdot l_3 = l_2 \cdot l_4 = 0$ and $l_1 \cdot l_3 = l_1 \cdot l_4 = 1$. Together with $l_3 \cdot l_4 = 1$, it implies that

$l_1 \cup l_3 \cup l_4$ forms a triangle. So n must be six and the triangle is anti-canonical. Hence $1 = -K \cdot l_2 = (l_1 + l_3 + l_4) \cdot l_2 = 0$ gives the contradiction.

If some of the l_i's are the same, the only possible way that $D \cdot R$ can be bigger than one is $l_2 = l_3$ (or l_4). In this case we have

$$\begin{aligned} D \cdot R &= (l_1 - l_2) \cdot (l_2 + l_4) \\ &= l_1 \cdot l_4 - l_2^2 - l_2 \cdot l_4 \\ &= l_1 \cdot l_4 + 1 - 1 \\ &= l_1 \cdot l_4 \leq 1. \end{aligned}$$

Hence we always have $D \cdot R \leq 1$. By replacing D by $-D$ if necessary, we obtain $|D \cdot R| \leq 1$ provided that $n \leq 6$. Hence the proposition. $\square$

The above equality as vector bundle decomposition of $L\mathcal{E}_n$ follows immediately from the proposition and the definition of $L\mathcal{D}_{n-1}$ and $\mathcal{S}^-$ when $n \leq 6$. In fact, such decomposition is a decomposition of $L\mathcal{E}_n$ as $L\mathcal{D}_{n-1}$-representation bundles. The decompositions for $L\mathcal{E}_7$ and $L\mathcal{E}_8$ will be discussed in later sections.

Reconstructing $L\mathcal{E}_n$ from $L\mathcal{D}_{n-1}$

We can also use the above vector bundle decompositions to reconstruct the fiberwise Lie algebra structure on $L\mathcal{E}_n$ from $L\mathcal{D}_{n-1}$. This provides us with an alternative way to describe the Lie algebra bundle $L\mathcal{E}_n$. When $n \leq 6$ the construction is completely analogous to the construction of $L\mathcal{E}_n$ from its decomposition under $L\mathcal{E}_{n-1}$. We omit the details here. The $n = 8$ case is different and it will be discussed in section eight.

4 $\mathcal{A}_{n-2}$-bundle over ruled surfaces with a section

4.1 Reduction to A_{n-2}-bundle over X_n using $\mathcal{S}^+$

We continue our notations from the D_{n-1}-bundle $\mathcal{D}_{n-1}$ over X_n. The Dynkin diagram of $A_{n-2} = \mathbf{sl}(n-1)$ can be obtained by removing the node in the Dynkin diagram of D_{n-1} which corresponds to the spinor representation S^+ (or S^- which we will discuss in the next subsection).

To reduce $\mathcal{D}_{n-1}$ to an A_{n-2}-bundle over X_n, we choose a line section S to the ruling R over X_n. This corresponds to a direct summand of $\mathcal{S}^+ = \bigoplus O_X(S)$. Each singular fiber of the ruling consists of two lines. One of them, say C, intersect the section S and it satisfies $C^2 = -1 = C \cdot K, C \cdot R = 0$ and $C \cdot S = 1$. Using these, we define the vector bundle

$$\Lambda_{n-2} = \bigoplus_{\substack{C^2=-1 \\ CK=-1 \\ CR=0 \\ CS=1}} O_{X_n}(C)$$

over X_n. Notice that the other line of the singular fiber containing C is linear equivalent to $-C + R$. Therefore we have

$$\mathcal{W}_{n-1} = \Lambda_{n-2} + \Lambda_{n-2}^* \otimes O_{X_n}(R).$$

In particular, the rank of Λ_{n-2} equals $n-1$. It is not difficult to check that there is an isomorphism

$$\det : \det \Lambda_{n-2} \xrightarrow{\cong} O_{X_n}(-K_X - 2S + (n-4)R).$$

Now the automorphism bundle of Λ_{n-2} preserving det is an A_{n-2}-bundle $\mathcal{A}_{n-2}$ over X_n. If we denote the lth-wedge product of Λ_{n-2} as Λ_{n-2}^l. For instance we have $\Lambda_{n-2}^0 = \mathbf{1}$ and $\Lambda_{n-2}^{n-1} = \det \Lambda_{n-2}$. Then Λ_{n-2}^l for $1 \le l \le n-2$ are representation bundles of $\mathcal{A}_{n-1}$ corresponding to all the fundamental representations of $A_{n-2} = \mathbf{sl}(n-1)$. We also have $L\mathcal{A}_{n-2} = End_0(\Lambda_{n-2})$.

We have the following numerical characterization of $L\mathcal{A}_{n-2}$, whose proof we similar to previous arguments and therefore skipped.

Proposition 23 *We have the following decomposition of $L\mathcal{A}_{n-2}$ over X_n :*

$$L\mathcal{A}_{n-2} = O^{\oplus n-2} + \bigoplus_{\substack{D^2=-2 \\ DK=0 \\ DR=0 \\ DS=0}} O(D)$$

Decompose $\mathcal{D}_{n-1}$-bundles under $\mathcal{A}_{n-2}$

From the previous isomorphism, we have $\Lambda_{n-2}^{*}=\Lambda_{n-2}^{n-2}\otimes\det\Lambda_{n-2}^{*}=\Lambda_{n-2}^{n-2}\otimes O\left(K+2S+(4-n)R\right)$. Using this, we get

$$\mathcal{W}_{n-1}=\Lambda_{n-2}+\Lambda_{n-2}^{n-2}\otimes O\left(K+2S+(5-n)R\right).$$

Therefore we get similar decompositions for wedge products of $\mathcal{W}_{n-1}$ in terms of fundamental representation bundles of $\mathcal{A}_{n-2}$. For example, using the isomorphisms $L\mathcal{D}_{n-1}=\Lambda^{2}\mathcal{W}_{n-1}\otimes O\left(-R\right)$ and $L\mathcal{A}_{n-2}=End_{0}\left(\Lambda_{n-2}\right)$, we have the following isomorphism between representation bundles of $L\mathcal{A}_{n-2}$:

$$L\mathcal{D}_{n-1}=L\mathcal{A}_{n-2}+O+\Lambda_{n-2}^{2}\otimes O\left(-R\right)+\left(\Lambda_{n-2}^{2}\right)^{*}\otimes O\left(R\right).$$

Besides wedge products of $\mathcal{W}_{n-1}$, there are two other fundamental representation bundles of $\mathcal{D}_{n-1}$, namely $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$. For them we have the following decompositions:

$$\mathcal{S}^{+}=\sum_{l=0}^{\left[\frac{n-1}{2}\right]}\Lambda_{n-2}^{2l}\otimes O\left(S-lR\right),$$

and

$$\mathcal{S}^{-}=\sum_{l=1}^{\left[\frac{n}{2}\right]}\Lambda_{n-2}^{2l-1}\otimes O\left(S-lR\right).$$

We leave the verification of these two formulae for our readers. These decompositions of representation bundles correspond to decomposition of representations of D_{n-1} under A_{n-2}:

$$\begin{aligned}
D_{n-1} &= A_{n-2}+\mathbf{1}+\Lambda_{n-2}^{2}+\left(\Lambda_{n-2}^{2}\right)^{*},\\
W_{n-1} &= \Lambda_{n-2}+\Lambda_{n-2}^{*}=\Lambda_{n-2}+\Lambda_{n-2}^{n-2},\\
S^{+} &= \sum_{l=0}^{\left[\frac{n-1}{2}\right]}\Lambda_{n-2}^{2l}\text{ and }S^{-}=\sum_{l=1}^{\left[\frac{n}{2}\right]}\Lambda_{n-2}^{2l-1}.
\end{aligned}$$

4.2 Reduction to A_{n-2}-bundle over X_n using $\mathcal{S}^{-}$

As we mentioned that the Dynkin diagram of $A_{n-2}=\mathbf{sl}\left(n-1\right)$ can also be obtained by removing the node in the Dynkin diagram of D_{n-1} which corresponds to the spinor representation S^{-}, we can also reduce $\mathcal{D}_{n-1}$ to an A_{n-2}-bundle over X_n by choosing a direct summand $O\left(T\right)$ of $\mathcal{S}^{-}$. This means that T satisfies $T^{2}=-2$, $T\cdot K=0$ and $T\cdot R=1$ or equivalently $T+R$ is a ruling section of the ruling determined by R.

Similar to the previous subsection, we consider the vector bundle

$$\bar{\Lambda}_{n-2}=\bigoplus_{\substack{C^{2}=-1\\ CK=-1\\ CR=0\\ CT=1}}O_{X_n}\left(C\right)$$

over X_n. Also we have

$$\mathcal{W}_{n-1} = \bar{\Lambda}_{n-2} + \bar{\Lambda}_{n-2}^{*} \otimes O_{X_n}(R).$$

In particular, the rank of $\bar{\Lambda}_{n-2}$ equals $n-1$. It is not difficult to check that there is an isomorphism

$$\det : \det \bar{\Lambda}_{n-2} \overset{\cong}{\to} O_{X_n}(-K_X - 2T + (n-5)R).$$

Now the automorphism bundle of $\bar{\Lambda}_{n-2}$ preserving det is a principal A_{n-2}-bundle $\overline{\mathcal{A}}_{n-2}$ over X_n. If we denote the lth-wedge product of $\bar{\Lambda}_{n-2}$ as $\bar{\Lambda}_{n-2}^{l}$. For instance we have $\bar{\Lambda}_{n-2}^{0} = \mathbf{1}$ and $\bar{\Lambda}_{n-2}^{n-1} = \det \bar{\Lambda}_{n-2}$. Then $\bar{\Lambda}_{n-2}^{l}$ for $1 \leq l \leq n-2$ are representation bundles of $\overline{\mathcal{A}}_{n-2}$ corresponding to all the fundamental representations of $A_{n-2} = \mathbf{sl}(n-1)$. From similar reasons as before, we have the following proposition.

Proposition 24 *We have the following decomposition of $L\overline{\mathcal{A}}_{n-2}$ over X_n :*

$$L\overline{\mathcal{A}}_{n-2} = O^{\oplus n-2} + \bigoplus_{\substack{D^2=-2 \\ DK=0 \\ DR=0 \\ DT=0}} O(D)$$

From the previous isomorphism, we have $\bar{\Lambda}_{n-2}^{*} = \bar{\Lambda}_{n-2}^{n-2} \otimes \det \bar{\Lambda}_{n-2}^{*} = \bar{\Lambda}_{n-2}^{n-2} \otimes O(K + 2T + (5-n)R)$. Using this, we get

$$\mathcal{W}_{n-1} = \bar{\Lambda}_{n-2} + \bar{\Lambda}_{n-2}^{n-2} \otimes O(K + 2T + (6-n)R).$$

Therefore we get similar decompositions for wedge products of $\mathcal{W}_{n-1}$ in terms of fundamental representation bundles of $\overline{\mathcal{A}}_{n-2}$. For example, we have the following isomorphism between representation bundles of $L\overline{\mathcal{A}}_{n-2}$:

$$L\mathcal{D}_{n-1} = L\overline{\mathcal{A}}_{n-2} + O + \bar{\Lambda}_{n-2}^{2} \otimes O(-R) + \left(\bar{\Lambda}_{n-2}^{2}\right)^{*} \otimes O(R).$$

Besides wedge products of $\mathcal{W}_{n-1}$, there are two other fundamental representation bundles of $\mathcal{D}_{n-1}$, namely $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$. For them we have the following decompositions:

$$\mathcal{S}^{+} = \sum_{l=1}^{\left[\frac{n}{2}\right]} \bar{\Lambda}_{n-2}^{2l-1} \otimes O(T - (l-1)R).$$

and

$$\mathcal{S}^{-} = \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \bar{\Lambda}_{n-2}^{2l} \otimes O(T - lR),$$

We leave the verification of these two formulae for our readers. These decompositions of representation bundles correspond to decomposition of representations of D_{n-1} under A_{n-2}:

$$\begin{aligned} D_{n-1} &= A_{n-2} + \mathbf{1} + \Lambda^2_{n-2} + \left(\Lambda^2_{n-2}\right)^*, \\ W_{n-1} &= \Lambda_{n-2} + \Lambda^*_{n-2} = \Lambda_{n-2} + \Lambda^{n-2}_{n-2}, \\ S^+ &= \sum_{l=0}^{\left[\frac{n-1}{2}\right]} \Lambda^{2l}_{n-2} \text{ and } S^- = \sum_{l=1}^{\left[\frac{n}{2}\right]} \Lambda^{2l-1}_{n-2}. \end{aligned}$$

5 Quartic surface X_5 in $\mathbb{P}^4$ and its E_5-bundle

In this section we study the surface X_5 obtained by blowing up $\mathbb{P}^2$ at five generic points. It is a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4$. In earlier section, we have constructed a E_5-bundle $L\mathcal{E}_5$ over X_5 and its associated bundles $\mathcal{L}_5$ and $\mathcal{R}_5$ using lines and rulings on X_5.

Recall that E_5 is a classical Lie algebra, namely $E_5 = D_5 = \mathbf{so}(10)$. Moreover the E_5 fundamental representation $\mathbf{R}_5$ is just the standard representation of $\mathbf{so}(10)$. Correspondingly the bundle $\mathcal{R}_5$ on X_5 has a fiberwise non-degenerate quadratic form

$$q_5 : \mathcal{R}_5 \otimes \mathcal{R}_5 \to O(-K).$$

We now describe this quadratic form on $\mathcal{R}_5$. If we choose two rulings R_1 and R_2 on X_5 and write each of them as a sum of two intersecting lines, i.e. $R_1 = l_1 + l_2$ and $R_2 = l_3 + l_4$ with $l_1 \cdot l_2 = l_3 \cdot l_4 = 1$. In general we have $R_1 \cdot R_2 \leq 2$. When the equality sign holds, then these four lines must form a rectangle on X_5 because there is no triangle on X_n when $n < 6$. Moreover every rectangle on X_5 is an anti-canonical divisor. This determines q_5 on $O(R_1) \otimes O(R_2)$. Then the quadratic form q_5 on the whole $\mathcal{R}_5$ is obtained by extending these linearly.

The automorphism bundle of $\mathcal{R}_5$ preserving q_5 is a E_5 bundle over X_5 which was denoted $\mathcal{E}_5$ in the introduction. Moreover the associated Lie algebra of $\mathcal{E}_5$ is just $L\mathcal{E}_5$. If we regard $L\mathcal{E}_5$ as a Lie algebra bundle of type $\mathbf{so}(10)$, then $\mathcal{L}_5$ is its representation bundle corresponding to a spinor representation of $\mathbf{so}(10)$, which has rank equals 16.

If we fix a ruling R on X_5, then we break the symmetry from E_5 to $D_4 = \mathbf{so}(8)$. Their corresponding representation bundles $\mathcal{W}_4$, $\mathcal{S}^+$ and $\mathcal{S}^-$ have been discussed in section three. Nevertheless, there is an additional homomorphism among them.

$$f : \mathcal{S}^+ \otimes \mathcal{S}^- \otimes \mathcal{W}_4 \to O_{X_5}(-K - R).$$

This homomorphism f is constructed via isomorphisms $O_X(-K_X - R)$ and tensor products of various direct summands of $\mathcal{S}^+, \mathcal{S}^-$ and $\mathcal{W}_4$. For instance, using notations in the example of D_4 in last section, $O(H - L_1 - L_2), O(L_0 - L_3)$ and $O(H - L_0 - L_4)$ are direct summands of $\mathcal{S}^+, \mathcal{S}^-$ and $\mathcal{W}_4$ respectively. We have isomorphism

$$O(H - L_1 - L_2) \otimes O(L_0 - L_3) \otimes O(H - L_0 - L_4) \cong O(-K - R)$$

where $R = H - L_0$. We expect that this homomorphism f would be used to describe triality among these bundles in a manner similar to the triality for the Lie algebra $\mathbf{so}(8)$.

5.1 Reduction to $\mathbf{sl}(4) \times \mathbf{sl}(2) \times \mathbf{sl}(2)$-bundle

In this subsection, we discuss a degeneration of X_5 into nonnormal del Pezzo surface which is a normal crossing variety. In terms of $L\mathcal{E}_5$, it corresponds to

reducing the structure group from $E_5 = \mathbf{so}(10)$ to $A_3 \times A_1 \times A_1 = \mathbf{sl}(4) \times \mathbf{sl}(2) \times \mathbf{sl}(2) = \mathbf{so}(6) \times \mathbf{so}(4)$. Recall that X_5 is a complete intersection of two quadric hypersurfaces Q and Q' in $\mathbb{P}^4$. Now we degenerate one of these quadric hypersurfaces, say Q', into a union of two hyperplanes $H_1 \cup H_2$ and denote the degenerating family by $Q'(t)$ with $Q'(0) = H_1 \cup H_2$. We define the corresponding family of del Pezzo surfaces as $X_5(t) = Q \cap Q'(t)$ **.** So $X_5(0) = Q_{(1)} \cup Q_{(2)}$ where $Q_{(i)} = Q \cap H_i$. Each $Q_{(i)}$ is a quadric surface in $\mathbb{P}^3 \simeq H_i$, so it has two rulings by lines in $H_i \subset \mathbb{P}^4$.

We want to understand what happens to these lines on $X_5(t)$ as t approaches zero. The first step is to see which members of the two rulings on $Q_{(i)}$ can be deformed to lines in $X_5(t)$ for small nonzero t. Let C be the singular locus of $X_5(0)$. Then C is a conic curve $C = Q \cap H_1 \cap H_2$ inside $H_1 \cap H_2 \simeq \mathbb{P}^2$. As we vary t away from zero, $Q'(t)$ intersects C at four points[4] infinitesimally. We call them q_1, q_2, q_3 and q_4. More precisely, if we let $Z(t)$ be the intersection of C with $Q'(t)$ for nonzero t. Then $Z \triangleq \lim_{t\to 0} Z(t) = \{q_1, q_2, q_3, q_4\}$. Notice that $Z \subset Q_{(1)} \cap Q_{(2)} = C$. Now it is not difficult to see that a line in $Q_{(1)} \cup Q_{(2)}$ has a first order deformation inside $X_5(t)$ if and only if it passes through one of the q_i's. In fact, all these infinitesimal deformations are unobstructed and exhaust all possible lines in $X_5(t)$ for small nonzero t.

For example, there are two lines in $Q_{(1)}$ passes through each q_j which come from the two rulings on $Q_{(1)}$. The same count holds for $Q_{(2)}$. Therefore the number of lines on X_5 equals $4 \times 2 + 4 \times 2 = 16$. We want to obtain a corresponding decomposition for $\mathcal{L}_5(t)$ as t approach zero. On each of the $Q_{(i)}$'s, there is a rank two vector bundle $\mathcal{R}_{(i)}$ over it given by the two ruling on it. Namely

$$\mathcal{R}_{(i)} = \bigoplus_{\substack{R^2=0 \\ RK=-2}} O_{Q_{(i)}}(R).$$

This determines a $SL(2)$-bundle $\mathcal{A}_{(i)}$ over $Q_{(i)}$ because $\Lambda^2\mathcal{R}_{(i)} \cong K_{Q_{(i)}}^{-1/2}$. Now giving a ruling on $Q_{(1)}$ determined by R, a line in this ruling correspond to the zero set of a section of $O_{Q_{(1)}}(R)$. Such line passes through q_j if and only if the corresponding section can be lift to $\mathfrak{I}_{\{q_j\}} \otimes O_{Q_{(1)}}(R)$. Hence it is natural to define

$$\mathcal{L}_5(0) = \mathfrak{I} \otimes \mathcal{R}_{(1)} + \mathfrak{I} \otimes \mathcal{R}_{(2)},$$

with $\mathfrak{I} = \bigoplus_{q\in Z} \mathfrak{I}_{\{q\}}$ a rank four coherent sheaf on $X_5(0)$. One should compare this decomposition of $\mathcal{L}_5(0)$ with the decomposition of the E_5 representation $\mathbf{L}_5$ when we view it as a representation of $A_3 \times A_1 \times A_1$:

$$\mathbf{L}_5|_{A_3\times A_1\times A_1} = \Lambda_3 \otimes \Lambda_1 \otimes \mathbf{1} + \Lambda_3^* \otimes \mathbf{1} \otimes \Lambda_1.$$

[4]It should really be a zero dimensional scheme of length four. For simplicity we assume that Q is generic so that the four points are distinct.

Here Λ_n denotes the standard representation of $A_n = \mathbf{sl}(n)$. It would be useful to know exactly how $\mathcal{L}_5(t)$ degenerates as t goes to zero and compare the limit with $\mathcal{L}_5(0)$.

Next we want to understand the behavior of $\mathcal{R}_5(t)$ as t approach zero. Equivalently we want to study the behaviors of rulings on $X_5(t)$ as t approach zero. In Lie algebra term, this will correspond to the decomposition of E_5 representation $\mathbf{R}_5$ when we view it as a representation of $A_3 \times A_1 \times A_1$:

$$\mathbf{R}_5|_{A_3 \times A_1 \times A_1} = \Lambda_3^2 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \Lambda_1 \otimes \Lambda_1.$$

Here Λ_n^2 denotes the fundamental representation of A_n which is the wedge product of Λ_n with itself.

Now if we fix a ruling on $Q_{(1)}$ and a ruling on $Q_{(2)}$, then the sum of the corresponding lines passes through any point on $C = Q_{(1)} \cap Q_{(2)}$ can be deformed to a smooth rational curve R on $X_5(t)$ for small nonzero t. Moreover R satisfies $R^2 = 0$ and $R \cdot K_{X_5(t)} = -2$ and therefore it defines a ruling on $X_5(t)$.

On the other hand, if we fix two of the q_i's, say q_1 and q_2, then they define another type of ruling on nearby $X_5(t)$. To see this, we pick one of the $Q_{(i)}$'s, say $Q_{(1)}$. We choose the line in $Q_{(1)}$ that passes through q_1 in one of its ruling and choose another line in $Q_{(1)}$ that passes through q_2 in the other ruling. Then their sum can be deformed to a smooth rational curve on $X_5(t)$ for small nonzero t which determines a ruling on $X_5(t)$ as before. It is not difficult to show that this ruling on $X_5(t)$ is independent of the choice of $Q_{(i)}$ and the choice of the particular ruling on $Q_{(i)}$ that gives the line through q_1. Namely this ruling on $X_5(t)$ depends only on the choice of the two points among the q_i's.

One can also verify that this exhausts all possible ruling in a nearby $X_5(t)$. Notice that there are also four pairs of lines that determines a particular ruling on X_5.

Therefore we define a coherent sheaf on $X_5(0)$ as

$$\mathcal{R}_5(0) = \mathfrak{I} \wedge \mathfrak{I} + \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)}.$$

It would be interesting to know if $\mathcal{R}_5(t)$ does degenerate to $\mathcal{R}_5(0)$ as t approaches zero.

The above degeneration of X_5 into union of two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ can be described from a different viewpoint: If X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$

$$\delta : X \to \mathbb{P}^1 \times \mathbb{P}^1,$$

branched along a smooth curve B of bidegree (2,2). Then we have (See e.g [BPV])

$$\begin{aligned} K_X &= \delta^* \left(K_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes O(1,1) \right) \\ &= \delta^* O(1,1). \end{aligned}$$

So K^{-1} is ample and $K^2 = 4$. That is X is a blowup of $\mathbb{P}^2$ at five points, X_5. On the other hand, requiring a degree four del Pezzo surface in $\mathbb{P}^4$ to be a double

cover of $\mathbb{P}^1 \times \mathbb{P}^1$ is a codimension one condition, by dimension counting. Each ruling on $\mathbb{P}^1 \times \mathbb{P}^1$ gives X a ruling. Moreover $\delta^{-1}\left(\mathbb{P}^1 \times p\right)$ is a union of two lines on X if and only in $\mathbb{P}^1 \times p$ passes through a branched point of the double cover $B \xrightarrow{\iota} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr_2} \mathbb{P}^1$, where the first morphisms is the inclusion and the second morphism is the projection to the second factor. Since B is a genus one curve by the adjunction formula, $B \xrightarrow{pr_2 \circ \iota} \mathbb{P}^1$ has four branched points. This gives rise to eight lines on X. The other eight lines on X comes from considering the other ruling on $\mathbb{P}^1 \times \mathbb{P}^1$.

Now we degenerate B to a double curve. For example $B = 2\Delta$ where $\left\{\Delta = (x, x) \in \mathbb{P}^1 \times \mathbb{P}^1 : x \in \mathbb{P}^1\right\}$ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. Then X becomes a union of two copies of $\mathbb{P}^1 \times \mathbb{P}^1$, identified along their diagonals. In the degeneration of B to 2Δ, an infinitesimal near B will intersect Δ at four points q_1, q_2, q_3 and q_4. There are distinct points on Δ if the degeneration is generic. In fact, these points are the limit of the branched points of the double cover of B to $\mathbb{P}^1$ for either of the two rulings. Now the appearance of the reduction to $\mathbf{sl}(4) \times \mathbf{sl}(2) \times \mathbf{sl}(2)$ is clear.

Remark 25 *Recall that X_5 is the intersection of two quadric hypersurfaces Q and Q' in $\mathbb{P}^4$. Instead of degenerating one of them to hyperplanes, we can also degenerate both of them to union of hyperplanes and study the configuration of line and ruling under such degeneration. This degeneration of X_5 corresponds to the reduction of E_5 to $A_1 \times A_1 \times A_1 \times A_1$.*

6 Cubic surface X_6 and its E_6-bundle

When $n = 6$ then X_6 is a cubic surface in $\mathbb{P}^3$. Studying of lines on cubic surface has a long history in algebraic geometry. As we mentioned earlier, every line on X_6 determines a ruling and vice versa. Since once a line L on X_6 is chosen then we have the decomposition $\mathcal{L}_6 = \pi^*\mathcal{L}_5 + \pi^*\mathcal{R}_5 \otimes O(-L) + O(L)$ from section two. The component $\pi^*\mathcal{R}_5 \otimes O(-L)$ corresponds to those lines intersecting L. However $E_5 = D_5 = \mathbf{so}(10)$ and $\mathbf{R}_5$ corresponds to the standard representation of $\mathbf{so}(10)$ and therefore carries a natural quadratic form as described in section five. There are five pairs of intersecting lines that intersect L on X_6. That is L constitutes one side of five different triangles. Remaining sides of any of these five triangles determines a ruling on X_6 and this ruling is independent of the choice of such triangle. As vector bundles we have isomorphism $\mathcal{R}_6 \cong \mathcal{L}_6^* \otimes O(-K)$. In terms of representations of E_6, we have $\mathbf{R}_6 = \mathbf{L}_6^*$ which is given by the outer-automorphism of E_6 responding to the $\mathbb{Z}/\mathbb{Z}_2$ symmetry of its Dynkin diagram.

Recall that we have representation bundles homomorphism

$$c_6 : \mathcal{L}_6 \otimes \mathcal{L}_6 \to \mathcal{R}_6$$

which exists for any n. On X_6 there is another representation bundle homomorphism

$$c_6^* : \mathcal{R}_6 \otimes \mathcal{R}_6 \to \mathcal{L}_6 \otimes O(-K).$$

This is because $\mathcal{R}_6$ is in fact a direct summand of $\mathcal{L}_6 \otimes \mathcal{L}_6$. The homomorphism c_6^* is simply the inclusion of $\mathcal{R}_6$ inside $\mathcal{L}_6 \otimes \mathcal{L}_6$ using the isomorphism $\mathcal{R}_6 \cong \mathcal{L}_6^* \otimes O(-K)$.

As we mentioned before, the automorphism bundle of the pair $(\mathcal{L}_6, \mathcal{R}_6)$ preserving c_6 and c_6^* is a bundle of type E_6 over X_6 which was denoted as $\mathcal{E}_6$. Moreover it associated Lie algebra bundle is precisely $L\mathcal{E}_6$. This follows from a direct construction of E_6 using corresponding products on $\mathbf{L}_6$ and $\mathbf{R}_6$ (see e.g. [Adams]).

If we use the isomorphism $\mathcal{R}_6 \cong \mathcal{L}_6^* \otimes O(-K)$ on the image of the homomorphism c_6. Then we obtain a triple product

$$\mathcal{L}_6 \otimes \mathcal{L}_6 \otimes \mathcal{L}_6 \to O(-K).$$

This triple product is symmetry in all three variables. In terms of the geometry of the cubic surface, this product can be described by determining whether three given lines on X_6 forms a triangle or not. Similarly we also have

$$\mathcal{R}_6 \otimes \mathcal{R}_6 \otimes \mathcal{R}_6 \to O(-2K).$$

6.1 Reduction to $\mathbf{sl}(3) \times \mathbf{sl}(3) \times \mathbf{sl}(3)$ bundle

In this subsection we degenerate the cubic surface into union of three generic planes in $\mathbb{P}^3$. We are going to see that the symmetry group of $\mathcal{L}_6, \mathcal{R}_6$ and $L\mathcal{E}_6$ would be reduced from E_6 to $\mathbf{sl}(3) \times \mathbf{sl}(3) \times \mathbf{sl}(3) = A_2 \times A_2 \times A_2$.

Let $X(t)$ be the family of cubic surfaces in $\mathbb{P}^3$ parametrized by t. We assume that $X(0)$ is a union of three planes $X(0) = H_1 \cup H_2 \cup H_3$ and $X(t)$ is smooth for nonzero t.Genericity of H_i means that their common intersection consists of a single point. For example if $f(z_0, z_1, z_2, z_3)$ is a generic homogenerous cubic polynomial, then

$$X(t) = \{tf(z_0, z_1, z_2, z_3) + z_1 z_2 z_3 = 0\} \subset \mathbb{P}^3,$$

would satisfy our assumption. We denote $C_i = H_j \cap H_k$ with $\{i, j, k\} = \{1, 2, 3\}$. As we various t, $X(t)$ intersects C_i at three points infinitesimally. We call them $Z_i = \{p_{i1}, p_{i2}, p_{i3}\}$. Then a line in H_3 which can be deformed to one in $X(t)$ for small t if and only if it intersects C_1 and C_2 at $p_{1\alpha}$ and $p_{2\beta}$ respectively, for some $\alpha, \beta \in \{1, 2, 3\}$. We denote it as $\overline{p_{1\alpha} p_{2\beta}}$. The number of such lines in H_3 equals $9 = 3 \times 3$. If we replace H_3 by either H_1 or H_2 then analogous statements would hold true. Conversely any one parameter family of lines on $X(t)$ parametrized by t would converge to one of these lines (see [Segre]). The total number of these lines equals $27 = 3 \times 3 + 3 \times 3 + 3 \times 3$.

In fact the decomposition of the representation $\mathbf{L}_6$ of E_6 under $A_2 \times A_2 \times A_2$ has the similar structure. If we denote the standard representation of A_2 by Λ_2 and its dual representation by Λ_2^*. Then

$$\mathbf{L}_6|_{A_2 \times A_2 \times A_2} = \Lambda_2 \otimes \Lambda_2^* \otimes \mathbf{1} + \mathbf{1} \otimes \Lambda_2 \otimes \Lambda_2^* + \Lambda_2^* \otimes \mathbf{1} \otimes \Lambda_2.$$

Before we discuss $\mathcal{L}_6(0)$ we need to define the coherent sheaf $\mathcal{I}_{(i)} = \oplus_{p \in Z_i} \mathcal{I}_{\{p\}}$ where $\mathcal{I}_{\{p\}}$ is the ideal sheaf of $\{p\}$ in $X_6(0)$. We will use the homomorphism $\mathcal{I}_{(i)} \otimes \mathcal{I}_{(i)} \to O_{X_6(0)}$ defined by $(\oplus_{p \in Z_i} s_p) \otimes \left(\oplus_{p \in Z_i} s'_p\right) \mapsto \sum_{p \in Z_i} s_p s'_p$.

Also $O_{H_i}(1)$ denotes the hyperplane bundle on H_i and we treat it as a rank one coherent sheaf on $X_6(0)$ whose restriction on $X_6(0) \setminus H_i$ is trivial. Now sections of $\mathcal{I}_{(1)} \otimes \mathcal{I}_{(2)} \otimes O_{H_3}(1)$ would correspond to line on H_3 meeting C_1 and C_2 at $p_{1\alpha}$ and $p_{2\beta}$ respectively, for some $\alpha, \beta \in \{1, 2, 3\}$. Therefore it is reasonable to propose that

$$\mathcal{L}_6(0) = \mathcal{I}_{(1)} \otimes \mathcal{I}_{(2)} \otimes O_{H_3}(1) + \mathcal{I}_{(2)} \otimes \mathcal{I}_{(3)} \otimes O_{H_1}(1) + \mathcal{I}_{(3)} \otimes \mathcal{I}_{(1)} \otimes O_{H_2}(1).$$

Next we discuss the triple product on $\mathcal{L}_6(0)$,

$$\mathcal{L}_6(0) \otimes \mathcal{L}_6(0) \otimes \mathcal{L}_6(0) \to O_{X_6(0)}(1).$$

Here $O_{X_6(0)}(1)$ can be interpreted as the anti-canonical sheaf on $X_6(0)$ even though $X_6(0)$ is nonnormal. Using the homomorphism $\mathcal{I}_{(2)} \otimes \mathcal{I}_{(2)} \to O_{X_6(0)}$ defined above, we obtain a homomorphism

$$\left(\mathcal{I}_{(1)} \otimes \mathcal{I}_{(2)} \otimes O_{H_3}(1)\right) \otimes \left(\mathcal{I}_{(2)} \otimes \mathcal{I}_{(3)} \otimes O_{H_1}(1)\right) \to \mathcal{I}_{(1)} \otimes \mathcal{I}_{(3)} \otimes O_{H_1}(1) \otimes O_{H_1}(1).$$

We can use the same homomorphism with 3 (and 1) replacing 2 to further compose with $\mathcal{I}_{(3)} \otimes \mathcal{I}_{(1)} \otimes O_{H_2}(1)$ and obtain a homomorphism from the tensor

product of the three components of $\mathcal{L}_6(0)$ into $O_{H_1}(1) \otimes O_{H_1}(1) \otimes O_{H_2}(1) = O_{X_6(0)}(1)$. This induces the triple product on $\mathcal{L}_6(0)$ in terms of its individual components. Of course, this is complete analogous to the description of the triple product on $\mathbf{L}_6$,

$$\mathbf{L}_6 \otimes \mathbf{L}_6 \otimes \mathbf{L}_6 \to \mathbb{C}.$$

in terms of its decomposition as representation of $A_2 \times A_2 \times A_2$:

$$\mathbf{L}_6|_{A_2 \times A_2 \times A_2} = \Lambda_2 \otimes \Lambda_2^* \otimes \mathbf{1} + \mathbf{1} \otimes \Lambda_2 \otimes \Lambda_2^* + \Lambda_2^* \otimes \mathbf{1} \otimes \Lambda_2.$$

In that case the triple product is given as follows: Let $s_1 \otimes t_2 \in \Lambda_2 \otimes \Lambda_2^* \otimes \mathbf{1} \subset \mathbf{L}_6$,$s_2 \otimes t_3 \in \mathbf{1} \otimes \Lambda_2 \otimes \Lambda_2^* \subset \mathbf{L}_6$ and $s_3 \otimes t_1 \in \Lambda_2^* \otimes \mathbf{1} \otimes \Lambda_2 \subset \mathbf{L}_6$. Then their product is

$$(s_1 \otimes t_2) \otimes (s_2 \otimes t_3) \otimes (s_3 \otimes t_1) \mapsto (t_1, s_1)(t_2, s_2)(t_3, s_3).$$

Here (t, s) is the natural pairing between Λ_2^* and Λ_2. And the product for all other combinations are zero except those obtained by permutation. (see [Segre])

Once the situation for $\mathcal{L}_6(0)$ is settled, the decomposition for $\mathcal{R}_6(0)$ is immediately. It is because if l is a line on X_6, then $R = -l - K$ determines a ruling on it. Hence

$$\begin{aligned} \mathcal{R}_6(0) &= \mathcal{I}_{(1)} \otimes \mathcal{I}_{(2)} \otimes O_{H_1}(1) \otimes O_{H_2}(1) \\ &\quad + \mathcal{I}_{(2)} \otimes \mathcal{I}_{(3)} \otimes O_{H_2}(1) \otimes O_{H_3}(1) \\ &\quad + \mathcal{I}_{(3)} \otimes \mathcal{I}_{(1)} \otimes O_{H_3}(1) \otimes O_{H_1}(1). \end{aligned}$$

The triple product on $\mathcal{R}_6(0)$ can be described in similar manner.

It would be useful to have a rigorous description of the degeneration of $\mathcal{L}_6(t)$ and $\mathcal{R}_6(t)$ as t goes to zero. Such description should also give the decomposition of $L\mathcal{E}_6(0)$. We recall that the decomposition of E_6 under $A_2 \times A_2 \times A_2$ is as follow:

$$E_6 = A_2 + A_2 + A_2 + \Lambda_2 \otimes \Lambda_2 \otimes \Lambda_2 + \Lambda_2^* \otimes \Lambda_2^* \otimes \Lambda_2^*.$$

The description of E_6 on $\mathbf{L}_6 = \Lambda_2 \otimes \Lambda_2^* \otimes \mathbf{1} + \mathbf{1} \otimes \Lambda_2 \otimes \Lambda_2^* + \Lambda_2^* \otimes \mathbf{1} \otimes \Lambda_2$ in terms of this decomposition is very simple. Each A_2 component in E_6 acts on the corresponding factor in each component of $\mathbf{L}_6$. To describe the action of $\Lambda_2 \otimes \Lambda_2 \otimes \Lambda_2$ on $\Lambda_2 \otimes \Lambda_2^* \otimes \mathbf{1}$, we first notice that there is a homomorphism from the tensor product of Λ_2 with itself to its dual space. It is because, being the standard representation of A_2, the third wedge power of Λ_2 is $\mathbf{1}$. Using this and the natural pairing between Λ_2 and Λ_2^*, we obtain the homomorphism

$$(\Lambda_2 \otimes \Lambda_2 \otimes \Lambda_2) \bigotimes (\Lambda_2 \otimes \Lambda_2^* \otimes \mathbf{1}) \to (\Lambda_2^* \otimes \mathbf{1} \otimes \Lambda_2),$$

which determines the action of $\Lambda_2 \otimes \Lambda_2 \otimes \Lambda_2$ on $\Lambda_2 \otimes \Lambda_2^* \otimes \mathbf{1}$. The action of $\Lambda_2 \otimes \Lambda_2 \otimes \Lambda_2$ on other components is similar. Also the action of $\Lambda_2^* \otimes \Lambda_2^* \otimes \Lambda_2^*$ on $\mathbf{L}_6$ is analogous. This gives the complete description of E_6 on $\mathbf{L}_6$ in this decomposition.

6.2 Reduction to $\mathbf{sl}(2) \times \mathbf{sl}(6)$ bundle

In this section we degenerate the cubic surface into union of a plane H and a smooth quadric surface Q in $\mathbb{P}^3$. We are going to see that the symmetry group of $\mathcal{L}_6, \mathcal{R}_6$ and $L\mathcal{E}_6$ would be reduced from E_6 to $\mathbf{sl}(6) \times \mathbf{sl}(2) = A_5 \times A_1$.

Let $X(t)$ be the family of cubic surfaces in $\mathbb{P}^3$ parametrized by t. We assume $X(0) = H \cup Q$ and $X(t)$ is smooth for nonzero t. We denote $C = H \cap Q$ and $Z = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ consists of those points where C intersects $X(t)$ infinitesimally as we various t away from zero.

Now a line in H which can be deformed to one in $X(t)$ for small t if and only if it intersects C at two of the six points in Z. On Q, there are two rulings. A member in one of the two rulings on Q can be deformed to a line in $X(t)$ for small t if and only if it intersects C at one of the six points in Z. Moreover any one parameter family of lines on $X(t)$ parametrized by t would converge to one of these lines. The total number of these lines equals $27 = \binom{6}{2} + 6 \times 2$.

In fact the decomposition of the representation $\mathbf{L}_6$ of E_6 under $A_5 \times A_1$ has the similar structure. If we denote the standard representation of A_n by Λ_n and the fundamental representation of A_n given by the second wedge product of its standard representation by Λ_n^2. Then

$$\mathbf{L}_6|_{A_5 \times A_1} = \Lambda_5^2 \otimes \mathbf{1} + \Lambda_5 \otimes \Lambda_2.$$

We define the coherent sheaf $\mathcal{I} = \oplus_{p \in Z} \mathcal{I}_{\{p\}}$ and we denote the hyperplane bundle on H by $O_H(1)$. We treat $O_H(1)$ as a rank one coherent sheaf on $X_6(0)$ whose restriction on $X_6(0) \backslash H$ being trivial. Moreover we define the rank two vector bundle

$$\mathcal{R}_Q = \bigoplus_{\substack{R^2=0 \\ RK_Q=-2}} O_Q(R),$$

on Q in terms of it rulings. We also treat $\mathcal{R}_Q$ as a rank two coherent sheaf on $X_6(0)$ whose restriction on $X_6(0) \backslash Q$ being trivial. Therefore it is reasonable to propose that

$$\mathcal{L}_6(0) = \Lambda^2 \mathcal{I} \otimes O_H(1) + \mathcal{I} \otimes \mathcal{R}_Q.$$

In order to describe the triple product on $\mathcal{L}_6(0)$, we consider several homomorphisms. The first one is a homomorphism from $\bigotimes^3 \left(\Lambda^2 \mathcal{I}\right)$ to $O_{X_6(0)}$ defined by the composition of naturally defined homomorphisms:

$$\bigotimes^3 \left(\Lambda^2 \mathcal{I}\right) \to \Lambda^3 \left(\Lambda^2 \mathcal{I}\right) \to \Lambda^6 \mathcal{I} = \mathcal{I}_Z \to O_{X_6(0)}.$$

The second one is a homomorphism from $\left(\Lambda^2 \mathcal{I}\right) \otimes \mathcal{I} \otimes \mathcal{I}$ to $O_{X_6(0)}$ defined by

$$\left(\oplus_{p \neq q \in Z} s_{pq}\right) \otimes \left(\oplus_{p \in Z} s'_p\right) \otimes \left(\oplus_{p \in Z} s''_p\right) \mapsto \sum_{p \neq q \in Z} s_{pq} \left(s'_p s''_q + s'_q s''_p\right).$$

The last one is the isomorphism between $\Lambda^2\mathcal{R}_Q$ and $O_Q(1)$. We leave the details for the construction of the triple product to our readers. The construction of $\mathcal{R}_6(0)$ and its triple product is also similar (see the previous section).

7 Branched cover X_7 of $\mathbb{P}^2$ along a quartic curve and its E_7-bundle

The quartic plane curve comes into our picture because every degree two Del Pezzo surface X is a double cover of $\mathbb{P}^2$ branched along a quartic plane curve C. It is a famous old fact that there are 28 bitangents to C. This can be seen either by analyzing the dual curve or by identifying each bitangent with an odd theta characteristic where there are 28 of them. ([Clemens])

7.1 Degree two del Pezzo surface and its E_7 bundle

The representation bundle $\mathcal{R}_7$

On X_7 we have $\mathcal{L}_7 = \bigoplus_{\substack{l^2=-1 \\ l\cdot K=-1}} O(l)$ and $\mathcal{R}_7 = O(-K)^{\oplus 7} + \bigoplus_{\substack{R^2=0 \\ R\cdot K=-2}} O(R)$. Notice that the extra summand $O(-K)^{\oplus 7}$ in $\mathcal{R}_7$ does not correspond to rulings on X_7. We now explain the appearance of this summand from two different points of view. First we consider the representation bundles homomorphism

$$\mathcal{L}_n \otimes \mathcal{L}_n \to \mathcal{R}_n.$$

For $n \leq 6$, the construction of this homomorphism is related to the fact that if l_1 and l_2 are distinct lines on X_6, then $l_1 + l_2$ determines a ruling on X_6 provided that $l_1 \cdot l_2 = 1$. The anti-canonical class of X_n is very ample and defines an embedding of X_n inside $\mathbb{P}^{9-n}$. Therefore $l_1 \cdot l_2$ is either zero or one. When $n = 7$, the anti-canonical line bundle is no longer very ample and its sections only defines X_7 as a double cover of $\mathbb{P}^2$. Therefore $l_1 \cdot l_2$ can be zero, one or two. Explicitly, if we represent X_7 as blowup of $\mathbb{P}^2$ at seven points with exceptional locus $\bigcup_{i=1}^7 L_i$. Then $l_1 \equiv 3H - \sum_{i=1}^7 L_i - L_j$ and $l_2 = L_j$ are two distinct lines on X_7 with $l_1 \cdot l_2 = 2$. When this happens, we have $O(l_1 + l_2) = O(-K)$ for $j = 1, ..., 7$. This explains why we need the extra factor $O(-K)^{\oplus 7}$ in $\mathcal{R}_7$.

Second point of view: Such an extra factor can also be explained in terms of Lie algebra theory. As a representation of E_7, the adjoint representation of E_7 coincides with $\mathbf{R}_7$. Therefore we expect $L\mathcal{E}_7$ and $\mathcal{R}_7$ to be equal up to tensoring with a line bundle over X_7. Recall that $L\mathcal{E}_7 = O_{X_7}^{\oplus 7} + O(D)$ where we sum over divisors D with $D^2 = -2$ and $D \cdot K = 0$. For any such D, we have $(D-K)^2 = 0$ and $(D-K)\cdot K = -2$ because $K^2 = 2$. This implies that $D - K$ determines a ruling on X_7. That is

$$L\mathcal{E}_7 \otimes O(-K) = \mathcal{R}_7.$$

So the $O_{X_7}^{\oplus 7}$ summand in $L\mathcal{E}_7$ turns into $O(-K)^{\oplus 7}$summand in $\mathcal{R}_7$.

Quadratic structure on $\mathcal{L}_7$

For any line l on X_7, using $K^2 = 2$, we have $(-l-K)^2 = -1$ and $(-l-K)\cdot K = -1$. That is, there is another line l' on X_7 which is linearly equivalent to

$-l-K$. Using isomorphisms $O(l) \otimes O(l') \xrightarrow{\cong} O(-K)$ with various line l, we obtain a fiberwise non-degenerate quadratic form on $\mathcal{L}_7$,

$$q_7 : \mathcal{L}_7 \otimes \mathcal{L}_7 \to O(-K).$$

This quadratic form is preserved by the action of $L\mathcal{E}_7$.

Notice that $l \cdot l' = l \cdot (-l-K) = 2$ and l' can be characterized as the only line with this property.

Proposition 26 *If l is a line on X_7, then there is exactly one line l' on X_7 such that $l \cdot l' = 2$. Moreover $l' \cong -l-K$.*

Proof of proposition: If l is a line on X_7, using $K^2 = 2$, we get $(-l-K)^2 = (-l-K) \cdot K = -1$. Hence there is a line l' on X_7 linearly equivalent to $-l-K$. Moreover $l \cdot l' = 2$.

On the other hand if l_1 is any line with $l \cdot l_1 = 2$, then

$$l_1 \cdot l' = l_1 \cdot (-l-K) = -1.$$

Since both l_1 and l' are irreducible effective divisors on X_7, their intersection must be nonnegative unless they are the same. So we get uniqueness result. $\square$

Because of this proposition, the set of 56 lines on X_7 is naturally divided into 28 pairs of lines. If we consider X_7 as a double cover of $\mathbb{P}^2$,

$$\delta : X_7 \to \mathbb{P}^2.$$

The branched locus B of δ is a quartic curve on $\mathbb{P}^2$. It is classically known that there are 28 bitangents to B. The inverse image of each such bitangent consists precisely a pair of lines l, l' on X_7 with $l \cdot l' = 2$. The above quadratic form q_7 is just reflecting this property because $l + l' = -K_X$ which is the pullback of the hyperplane bundle of $\mathbb{P}^2$ via δ: $K_X^{-1} = \delta^* (K_{\mathbb{P}^2} \otimes O(2))^{-1} = \delta^* O_{\mathbb{P}^2}(1)$.

Remark 27 *Consider the action of $L\mathcal{E}_7$ on $\mathcal{L}_7$ given by the homomorphism $L\mathcal{E}_7 \otimes \mathcal{L}_7 \to \mathcal{L}_7$. If we use the quadratic form q_7 to identify $\mathcal{L}_7$ with $\mathcal{L}_7^* \otimes O(-K)$ and the fact that $L\mathcal{E}_7 \cong (L\mathcal{E}_7)^*$, then the action of $L\mathcal{E}_7$ on $\mathcal{L}_7$ gives rise to $\mathcal{L}_7 \otimes \mathcal{L}_7 \to L\mathcal{E}_7 \otimes O(-K)$. Using the isomorphism between $L\mathcal{E}_7 \otimes O(-K)$ and $\mathcal{R}_7$, we obtain $\mathcal{L}_7 \otimes \mathcal{L}_7 \to \mathcal{R}_7$ which is just the homomorphism c_7.*

Quartic structure on $\mathcal{L}_7$

Next we want to describe a symmetric quartic form on $\mathcal{L}_7$:

$$f_7 : \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \to O(-2K)$$

which is invariant under the action of $L\mathcal{E}_7$. As we have mentioned in the introduction, (the identity component of) the bundle of automorphisms of $\mathcal{L}_7$ preserving q_7 and f_7 determined a E_7-bundle $\mathcal{E}_7$ over X_7 whose associated Lie algebra bundle is just $L\mathcal{E}_7$.

To understand this, we notice that this quartic form f_7 globalizing a corresponding quartic form f on $\mathbf{L}_7$ which is invariant under E_7 (see chapter 12 of [Adams]). Moreover the identity component of the group of automorphism of $\mathbf{L}_7$ preserving it and the Killing form is a simply connected Lie group of type E_7.

We are going to give two different descriptions of f_7.

First description: Given three distinct lines l_1, l_2 and l_3 on X_7, we have

$$(-l_1 - l_2 - l_3 - 2K) \cdot K = -1,$$

and

$$(-l_1 - l_2 - l_3 - 2K)^2 = -7 + 2\left(l_1 \cdot l_2 + l_2 \cdot l_3 + l_1 \cdot l_3\right).$$

Hence there is a line l_4 in the linearly equivalent class of $-l_1 - l_2 - l_3 - 2K$ if and only if $l_1 \cdot l_2 + l_2 \cdot l_3 + l_1 \cdot l_3 = 3$. We have $0 \leq l_i \cdot l_j \leq 2$ for $i \neq j$. So, up to permutation of their indexes, there are only two possibilities for us to have $l_1 \cdot l_2 + l_2 \cdot l_3 + l_1 \cdot l_3 = 3$. Namely, case (1): $l_1 \cdot l_2 = l_2 \cdot l_3 = l_1 \cdot l_3 = 1$ and case (2) $l_1 \cdot l_2 = 2, l_1 \cdot l_3 = 1, l_2 \cdot l_3 = 0$. In case (1), we also have $l_i \cdot l_4 = 1$ for $i = 1, 2, 3$. That is $l_i \cdot l_j = 1$ for all $i \neq j$ between 1 and 4. In case (2), we also have $l_3 \cdot l_4 = 2, l_2 \cdot l_4 = 1$ and $l_1 \cdot l_4 = 0$.

In either cases, we have isomorphism

$$O(l_1) \otimes O(l_2) \otimes O(l_3) \xrightarrow{\cong} O(-l_4) \otimes O(-2K),$$

or equivalently

$$O(l_1) \otimes O(l_2) \otimes O(l_3) \otimes O(l_4) \xrightarrow{\cong} O(-2K).$$

Combining all such homomorphisms with l_1, l_2, l_3 satisfying either (1) or (2) and with $l_4 \equiv -l_1 - l_2 - l_3 - 2K$, we obtain a homomorphism

$$f_7 : \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \to O(-2K).$$

By construction, it is clear that f_7 is symmetric. One can also verify directly that f_7 is invariant under the action of $L\mathcal{E}_7$. Moreover, on each fiber of $\mathcal{L}_7$, the restriction of f_7 to it is isomorphic to f on $\mathbf{L}_7$. Therefore (the identity component of) the bundle of automorphisms of $\mathcal{L}_7$ preserving the quadratic form q_7 and the quartic form f_7 is a principal bundle $\mathcal{E}_7$ of type E_7. This also implies that its associated Lie algebra bundle is just $L\mathcal{E}_7$ because of the fact that both q_7 and f_7 are invariant under the action of $L\mathcal{E}_7$.

Second description: Recall that, for any $n \leq 8$, we have a representation bundles homomorphism: $c_n : \mathcal{L}_n \otimes \mathcal{L}_n \to \mathcal{R}_n$. However, by the previous section, we also have an isomorphism $\mathcal{R}_7 = L\mathcal{E}_7 \otimes O(-K)$. Combining these two, we have the following homomorphism:

$$\mathcal{L}_7 \otimes \mathcal{L}_7 \to L\mathcal{E}_7 \otimes O(-K).$$

Also $\mathcal{L}_7$ is a representation bundle of $L\mathcal{E}_7$, so it gives another homomorphism

$$L\mathcal{E}_7 \otimes \mathcal{L}_7 \to \mathcal{L}_7.$$

On the other hand, there is an isomorphism

$$\mathcal{L}_7 \cong \mathcal{L}_7^* \otimes O(-K),$$

given by the quadratic form q_7. Combining these, we obtain a homomorphism

$$\mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \to \mathcal{L}_7^* \otimes O(-2K),$$

which is equivalent to the above quartic form

$$f_7 : \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \otimes \mathcal{L}_7 \to O(-2K).$$

We leave the verifications to our readers. In this description, it is not obvious that f_7 is symmetric.

7.2 Reduction to $\mathbf{so}(12) \times \mathbf{sl}(2)$-bundle

We recall that $\mathbf{so}(12) \times \mathbf{sl}(2) = D_6 \times A_1$.

Now we fix a ruling R on X_7. A line l on X_7 would have $l \cdot R$ equals zero, one or two. However, if $l \cdot R = 0$ then there is another line l' on X_7 with $l' \cdot R = 2$. In fact $l' \equiv l - R - K$ which can be verified directly using $K^2 = 2$. This give

$$\mathcal{L}_7 = \mathcal{S}^+ + \mathcal{W}_6 \otimes \Lambda_1.$$

Here Λ_1 is the rank two vector bundle $O_{X_7} + O_{X_7}(-R-K)$

$$\Lambda_1 = O_{X_7} + O_{X_7}(-R-K).$$

The automorphism bundle of Λ_1 preserving $\det : \det \Lambda_1 \xrightarrow{\cong} O(-R-K)$ is a $SL(2)$-bundle over X_7 which we call $\mathcal{A}_1^{X_7}$, or simply $\mathcal{A}_1$. Its corresponding $\mathbf{sl}(2)$-Lie algebra bundle will be called $L\mathcal{A}_1^{X_7}$, or simply $L\mathcal{A}_1$. Then the above decomposition is a decomposition of $L\mathcal{D}_6 + L\mathcal{A}_1$-representation bundles corresponding to the following decomposition of E_7 representation:

$$\mathbf{L}_7|_{D_6 \times A_1} = S^+ \otimes \mathbf{1} + W_6 \otimes \Lambda_1,$$

where Λ_1 is the standard representation of $A_1 = \mathbf{sl}(2)$.

Next we want to decompose $L\mathcal{E}_7$ under $L\mathcal{D}_6 + L\mathcal{A}_1$. First we have

$$\begin{aligned} L\mathcal{A}_1 &= End_0(O + O(-R-K)) \\ &= O(R+K) + O + O(-R-K). \end{aligned}$$

If we write $D = R + K$ or $-R-K$, then we have $D^2 = -2$ and $D \cdot K = 0$. That is $L\mathcal{A}_1$ is a vector subbundle of $L\mathcal{E}_7$. Recall that $\mathcal{S}^- = \bigoplus O(T)$ with T

satisfying $T^2 = -2, T \cdot K = 0$ and $T \cdot R = 1$, is always a subbundle of $L\mathcal{E}_7$. By direct computations, we have

$$\mathcal{S}^- \otimes O(R+K) = \bigoplus_{\substack{R^2=-2 \\ RK=0 \\ RR=-1}} O(R),$$

and therefore also a subbundle of $L\mathcal{E}_7$. Moreover

$$L\mathcal{E}_7 = L\mathcal{D}_6 + L\mathcal{A}_1 + \mathcal{S}^- + \mathcal{S}^- \otimes O(R+K).$$

As a decomposition of $L\mathcal{D}_6 + L\mathcal{A}_1$ representation bundles, we have

$$L\mathcal{E}_7 = L\mathcal{D}_6 + L\mathcal{A}_1 + \mathcal{S}^- \otimes \Lambda_1^*.$$

This corresponds to the following decomposition of E_7-representation under $D_6 \times A_1$:

$$E_7|_{D_6 \times A_1} = D_6 + A_1 + S^- \otimes \Lambda_1^{-1}.$$

We obtain a similar decomposition for $\mathcal{R}_7$ because $\mathcal{R}_7 = L\mathcal{E}_7 \otimes O(-K)$. Namely

$$\mathcal{R}_7 = O(R)\left(S^2\Lambda_1 + \mathcal{S}^- \otimes \Lambda_1 + \Lambda^2\mathcal{W}_6 \otimes (-K-2R)\right).$$

7.3 Reduction to sl (8) bundle

In this section we degenerate X_7 into a normal crossing variety which consists of two copies of $\mathbb{P}^2$ joining along a conic curve. We shall see that the E_7 structure on $\mathcal{L}_7$ and $\mathcal{R}_7$ reduces to $SL(8) = A_7$ structure under such degeneration.

To begin we consider X_7 as a double cover of $\mathbb{P}^2$ branched along a quartic plane curve. Now we want deform the quartic curve into a double conic. Let t be the deformation parameter and $B(t)$ be a family of quartic plane curve with $B(0) = 2C$ with C is smooth conic in $\mathbb{P}^2$. We also assume that $B(t)$ is smooth when t is not zero. Let us denote the double cover of $\mathbb{P}^2$ branched along $B(t)$ as

$$\delta(t) : X(t) \to \mathbb{P}^2.$$

In particular $X(0)$ consists of two copies of $\mathbb{P}^2$ joining along C. We call the two $\mathbb{P}^2$ as $\mathbb{P}^2_{(1)}$ and $\mathbb{P}^2_{(2)}$. We let $Z = \{p_1, p_2, ..., p_8\} \subset C$ be the set consisting of points where C meets $B(t)$ as we various t away from zero infinitesimally.

Now a line in $\mathbb{P}^2_{(i)}$ which can be deformed to one in $X(t)$ for small t if and only if it intersects C at two of the eight points in Z. Moreover any one parameter family of lines on $X(t)$ parametrized by t would converge to one of these lines. The total number of these lines equals

$$56 = \begin{pmatrix} 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 8 \\ 2 \end{pmatrix}.$$

In fact these 28 line on $\mathbb{P}^2$ corresponds precisely to the limit of the 28 bitangents to the quartic plane curve on $B(t)$ (see [Clemens]) We saw in section two, that if l is a line on X_7 then there is exactly one other line l' which intersect l at two points and moreover l' is in the linearly equivalent class of $-l-K$. This divides the 56 lines into 28 pairs. In our situation here, each pair consists of one line in $\mathbb{P}^2_{(1)}$ and one line in $\mathbb{P}^2_{(2)}$.

The decomposition of the representation $\mathbf{L}_7$ of E_7 under A_7 has the similar structure. If we denote the standard representation of A_n by Λ_n and the fundamental representation of A_n given by the kth wedge product of its standard representation by Λ_n^k. Then

$$\mathbf{L}_7|_{A_7} = \Lambda_7^2 + \Lambda_2^6.$$

We define the coherent sheaf $\mathcal{I} = \oplus_{p\in Z}\mathcal{I}_{\{p\}}$ and we denote the hyperplane bundle on $\mathbb{P}^2_{(i)}$ by $O_{\mathbb{P}^2_{(i)}}(1)$. We treat $O_{\mathbb{P}^2_{(i)}}(1)$ as a rank one coherent sheaf on $X(0)$ whose restriction on $X(0)\backslash\mathbb{P}^2_{(i)}$ being trivial. Therefore it is natural to define

$$\mathcal{L}_7(0) = \Lambda^2\mathcal{I}\otimes O_{\mathbb{P}^2_{(1)}}(1) + \Lambda^2\mathcal{I}\otimes O_{\mathbb{P}^2_{(2)}}(1).$$

We now also describe the quartic form on $\mathcal{L}_7(0)$,

$$f_7 : \mathcal{L}_7(0)\otimes\mathcal{L}_7(0)\otimes\mathcal{L}_7(0)\otimes\mathcal{L}_7(0)\to O(-2K).$$

In terms of lines on $X_7(0)$, nontrivial product in f_7 corresponds to either (1) four lines in one of the $\mathbb{P}^2_{(i)}$'s and they pass though all eight points of Z or (2) two pairs of lines.

As in other cases of degeneration into nonnormal surfaces, it is important to have a better understanding of the degeneration of $\mathcal{L}_7(t)$ (and $\mathcal{R}_7(t)$) and compare the limit with $\mathcal{L}_7(0)$.

8 E_8-bundles over X_8

Among exceptional Lie algebra, E_8 has many exceptional properties not share by other E_n's. These properties can also be seen on the geometry of X_8. For example the number of lines on X_8 equals 240 which is different from the dimension of $\mathbf{L}_8$ which equals 248. In fact

$$\mathcal{L}_8 = O(-K)^{\oplus 8} \bigoplus_{\substack{l^2=-1 \\ lK=-1}} O(l).$$

To explain this extra summand $O(-K)^{\oplus 8}$, we first notice that if l and l' are lines on X_8, then $(l+l'+K)^2 = -5 + 2l \cdot l'$ and $(l+l'+K)\cdot K = -1$. When $l \cdot l' = 2$, there is a line in the linearly equivalent class of $l+l'+K$. This suggests a product structure on $\bigoplus_{\substack{l^2=-1 \\ lk=-1}} O(l)$ and $\mathcal{L}_8$:

$$\mathcal{L}_8 \otimes \mathcal{L}_8 \to \mathcal{L}_8 \otimes O(-K)^{5}.$$

In fact $\mathcal{L}_8$ is isomorphic to the bundle $L\mathcal{E}_8$ up to tensoring with a line bundle:

$$\mathcal{L}_8 = L\mathcal{E}_8 \otimes O(-K).$$

Using $K^2 = 1$, we obtain $(l+K)^2 = -1$ and $(l+K)\cdot K = 0$. It implies the above isomorphism as vector bundles. In fact, one can check that they are isomorphic as representation bundles. In terms of representations of E_8, it corresponds simply to the fact that $\mathbf{L}_8$ coincides with the adjoint representation of E_8. This also explains the extra summand $O(-K)^{\oplus 8}$ in $\mathcal{L}_8$. The representation bundle $\mathcal{R}_8$ over X_8 is more complicated and we omit the discussion of it except to mention that its rank equal 3875.

Recall that, in the introduction of this paper, we discussed the seven types, (i),...,(vii), of lines on X_8 when we write X_8 as the blowup of $\mathbb{P}^2$ at eight points. These various types can be described in a more uniform way if we use the isomorphism $\mathcal{L}_8 = L\mathcal{E}_8 \otimes O(-K)$ and the earlier description of $L\mathcal{E}_8$ in terms of H and various L_i's. Here we denote the exceptional locus of the blowup as $L_1 \cup ... \cup L_8$ and the pullback of the hyperplane class of $\mathbb{P}^2$ as H. In the following table we display the various component types of $\mathcal{L}_8$ (except $O(-K)^{\oplus 8}$), the corresponding type of lines and the number of such lines.

$(L_i - L_j) - K$	(iv)	56			
$(H - L_i - L_j - L_k) - K$	(v)	56	$(-H + L_i + L_j + L_k) - K$	(iii)	56
$\left(2H - \sum_{m=1}^{6} L_{i_m}\right) - K$	(vi)	28	$\left(-2H + \sum_{m=1}^{6} L_{i_m}\right) - K$	(ii)	28
$\left(3H - \sum_{j=1}^{8} L_j - L_i\right) - K$	(vii)	8	$\left(-3H + \sum_{j=1}^{8} L_j + L_i\right) - K$	(i)	8

with

(i) $\pi(D) = p_i$
(ii) $\pi(D)$ is a line passes through p_i and p_j
(iii) $\pi(D)$ is a conic passes through five of the p_i's
(iv) $\pi(D)$ is a cubic passes through seven of the p_i's and with one being a double point
(v) $\pi(D)$ is a quartic passes through 8 of the p_i's and three being double points
(vi) $\pi(D)$ is a quintic passes through 8 of the p_i's and six being double points
(vii) $\pi(D)$ is a sextic passes through 8 of the p_i's
and seven being double points and one triple point.

8.1 Reduction to $E_7 \times \mathbf{sl}(2)$-bundle

If we fix a line L on X_8, then blowing down L gives us a morphism $\pi : X_8 \to X_7$ and the relationship between the E_8-bundle on X_8 and E_7-bundle on X_7 has been discussed in section two and we get

$$\begin{aligned} L\mathcal{E}_8 = \pi^* L\mathcal{E}_7 + O + \pi^*\mathcal{L}_7 \otimes O(-L) + \pi^*\mathcal{L}_7^* \otimes O(L) \\ + O(-K-L) + O(K+L). \end{aligned}$$

Now we consider the rank two vector bundle $\Lambda_1 = O + O(L+K)$ over X_8. Notice that $\Lambda_1 \otimes O(-K)$ is a subbundle of $\mathcal{L}_8$. Let $\mathcal{A}_1^{X_8}$, or simply $\mathcal{A}_1$, be the automorphism bundle of Λ_1 preserving $\det : \det \Lambda_1 \overset{\cong}{\to} O(L+K)$. Then $\mathcal{A}_1$ is an A_1-bundle (or $\mathbf{sl}(2)$-bundle). The corresponding Lie algebra bundle is simply

$$L\mathcal{A}_1 = O(-L-K) + O + O(L+K),$$

which is a Lie subalgebra bundle of $L\mathcal{E}_8$.

Next we use the quadratic form q_7 on $\mathcal{L}_7$:

$$q_7 : \mathcal{L}_7 \otimes \mathcal{L}_7 \to O_{X_7}(-K_{X_7}),$$

to identify $\mathcal{L}_7^*$ with $\mathcal{L}_7 \otimes O(K_{X_7})$. Using the adjunction formula $K_{X_8} = \pi^* K_{X_7} + L$, we obtain an isomorphism

$$\pi^*\mathcal{L}_7^* \otimes O(L) \cong \pi^*\mathcal{L}_7 \otimes O(K_{X_8}).$$

Hence

$$\begin{aligned} & \pi^*\mathcal{L}_7 \otimes O(-L) + \pi^*\mathcal{L}_7^* \otimes O(L) \\ \cong\ & \pi^*\mathcal{L}_7 \otimes O(-L) \otimes (O + O(L+K)) \\ \cong\ & \pi^*\mathcal{L}_7 \otimes O(-L) \otimes \Lambda_1. \end{aligned}$$

Combining these isomorphisms, we have

$$L\mathcal{E}_8 = \pi^* L\mathcal{E}_7 + \mathcal{A}_1 + \pi^*\mathcal{L}_7 \otimes O(-L) \otimes \Lambda_1.$$

Since $\mathcal{L}_8 = L\mathcal{E}_8 \otimes O(-K)$, a decomposition of $\mathcal{L}_8$ is equivalent to a decomposition of $L\mathcal{E}_8$. Namely

$$\mathcal{L}_8 = \pi^*\mathcal{L}_7 \otimes \Lambda_1^* + \pi^*\mathcal{R}_7 \otimes O(-L) + \mathcal{A}_1 \otimes O(-K).$$

8.2 Reduction to D_8-bundle

Again, a decomposition of $\mathcal{L}_8$ is equivalent to a decomposition of $L\mathcal{E}_8$ because $\mathcal{L}_8 = L\mathcal{E}_8 \otimes O(-K)$,

Recall that there is a D_{n-1}-bundle over X_n for any given choice of ruling on X_n. On X_8, we have a D_8-bundle instead of a D_7-bundle over it. To construct this D_8-bundle $\mathcal{D}_8$ over X_8, we need to choose eight disjoint lines on X_8. We denote them as $L_1, ..., L_8$. Notice that such a choice is equivalent to represents X_8 as the blowup of $\mathbb{P}^2$ at eight distinct points. Let H be the pullback of the hyperplane bundle on $\mathbb{P}^2$ to X_8. We define the following rank sixteen vector bundle on X_8:

$$\mathcal{W}_8 = \bigoplus_{i=1}^{8} \left(O(L_i) + O(-L_i - K - H)\right).$$

The isomorphism $O(L_i) \otimes O(-L_i - K - H) \cong O(-K - H)$ defines a fiberwise quadratic form on $\mathcal{W}_8$:

$$\mathcal{W}_8 \otimes \mathcal{W}_8 \to O(-K - H).$$

Now we define the D_8-bundle $\mathcal{D}_8$ on X_8 as the automorphism bundle of $\mathcal{W}_8$ which preserves the above fiberwise quadratic form. We denote its associated Lie algebra bundle as $L\mathcal{D}_8$ which is itself a vector bundle of rank 120. Explicitly we have

$$L\mathcal{D}_8 = \Lambda^2 \mathcal{W}_8 \otimes O(K + H).$$

It is not difficult to check the following equality:

$$L\mathcal{D}_8 = O^{\oplus 8} + \bigoplus_{\substack{D^2=-2 \\ DK=0 \\ DH \text{ even}}} O(D).$$

In particular $L\mathcal{D}_8$ is a vector subbundle of

$$\mathcal{L}_8 \otimes O(-K) = L\mathcal{E}_8 = O^{\oplus 8} + \bigoplus_{\substack{D^2=-2 \\ DK=0}} O(D).$$

In fact $L\mathcal{D}_8$ is a Lie sub-algebra bundle of $L\mathcal{E}_8$.

Next we introduce the positive spinor bundle for $L\mathcal{D}_8$. We consider the vector bundle

$$\mathcal{S}^+ = \bigoplus_{\substack{l^2=-1 \\ lK=-1 \\ lH \text{ even}}} O(l),$$

which is a vector subbundle of $\mathcal{L}_8$. To see that $\mathcal{S}^+$ is a representation bundle of $L\mathcal{D}_8$ we need a homomorphism

$$L\mathcal{D}_8 \otimes \mathcal{S}^+ \to \mathcal{S}^+.$$

If a divisor D satisfies $D^2 = -2, DK = 0$ and a line l with $l \cdot H$ even, then the divisor $l' = D + l$ satisfies (i) $l'^2 = -3 + 2D \cdot l$, (ii) $l' \cdot K = -1$ and (iii) $l' \cdot H$ is even. Hence l' determines a line if and only if $D \cdot l = 1$. Using these various homomorphisms $O(D) \otimes O(l) \to O(l')$, we obtain the above homomorphism $L\mathcal{D}_8 \otimes \mathcal{S}^+ \to \mathcal{S}^+$ which makes $\mathcal{S}^+$ a representation bundle of $L\mathcal{D}_8$.

It is easy to see that the rank of $\mathcal{S}^+$ equals 128 and the above homomorphism corresponds to a spinor representation of D_8. Now $L\mathcal{D}_8 \otimes O(-K)$ and $\mathcal{S}^+$ are two subbundle of $\mathcal{L}_8$ with trivial intersection. Moreover

$$rank\mathcal{L}_8 = 248 = 120 + 128 = rank\left(L\mathcal{D}_8 \otimes O(-K)\right) + rank\mathcal{S}^+.$$

Therefore $\mathcal{L}_8$ is the direct sum bundle of $L\mathcal{D}_8 \otimes O(-K)$ and $\mathcal{S}^+$,

$$\mathcal{L}_8 = L\mathcal{D}_8 \otimes O(-K) + \mathcal{S}^+.$$

Similarly we have

$$L\mathcal{E}_8 = L\mathcal{D}_8 + \mathcal{S}^+ \otimes O(K).$$

The above two decompositions are decompositions as representation bundle of $L\mathcal{D}_8$ corresponding to

$$\mathbf{L}_8 = E_8 = D_8 + S^+,$$

for the Lie algebra $D_8 = \mathbf{so}(16)$.

8.3 Construction of Lie algebra bundle $L\mathcal{E}_n$ revisited

Now we present a different approach to describe the fiberwise Lie algebra structure on $L\mathcal{E}_n$. Since E_8 is the biggest exceptional Lie algebra, we will first use the decomposition $L\mathcal{E}_8 = L\mathcal{D}_8 + \mathcal{S}^+ \otimes O(K)$ to describe the Lie algebra structure on $L\mathcal{E}_8$. Then we can use this to describe $L\mathcal{E}_n$ with $n < 8$.

Before we begin, we need two homomorphisms between representation bundles of $L\mathcal{D}_8$. The first one is a homomorphism

$$\mathcal{S}^+ \otimes \mathcal{S}^+ \to O(-2K).$$

To describe this homomorphism, we notice that if l is a line on X_8 with $l \cdot H$ even, then $(-l - 2K)^2 = -1 = (-l - 2K) \cdot K$. Namely there is a line l' in the class $-l - 2K$. Moreover $l' \cdot H = -l \cdot H - 2K \cdot H$ is also even. That is $O(l')$ is a subbundle of $\mathcal{S}^+$. We then define the homomorphism $\mathcal{S}^+ \otimes \mathcal{S}^+ \to O(-2K)$ using these various isomorphism $O(l) \otimes O(l') \to O(-2K)$.

Notice that l' can also be characterized as the unique line on X_8 which intersects l at three points. Therefore the above homomorphism also gives

a fiberwise non-degenerate quadratic form on $\mathcal{S}^+$ and gives an isomorphism between $\mathcal{S}^+$ and $(\mathcal{S}^+)^* \otimes O(-2K)$.

The second homomorphism is an *anti-symmetric* bilinear homomorphism:

$$\mathcal{S}^+ \otimes \mathcal{S}^+ \to L\mathcal{D}_8 \otimes O(-2K).$$

If l and l' are two lines on X_8 with $l \cdot H$ and $l' \cdot H$ even. Then $D = l + l' + 2K$ satisfies (i) $D^2 = -6+2l\cdot l'$,(ii) $D\cdot K = 0$ and (iii) $D\cdot H$ is even. Therefore $O(D)$ is a subbundle of $L\mathcal{D}_8$ if and only if $l \cdot l' = 2$. Using these various isomorphisms $O(l)\otimes O(l')\otimes O(2K) \to O(D)$, we obtain the above homomorphism up to sign. To avoid the sign problem, we use a different description. We rewrite the spinor representation $L\mathcal{D}_8 \otimes \mathcal{S}^+ \to \mathcal{S}^+$ as the homomorphism $(\mathcal{S}^+)^* \otimes \mathcal{S}^+ \to L\mathcal{D}_8$ since $L\mathcal{D}_8$ is self-dual. Now we use the non-degenerate pairing $\mathcal{S}^+ \otimes \mathcal{S}^+ \to O(-2K)$ to identify $(\mathcal{S}^+)^*$ with $\mathcal{S}^+ \otimes O(2K)$, then we obtain $\mathcal{S}^+ \otimes \mathcal{S}^+ \to L\mathcal{D}_8 \otimes O(-2K)$ the anti-symmetric homomorphism which is invariant under $L\mathcal{D}_8$.

Now we can use the Lie algebra structure on $L\mathcal{D}_8$

$$\alpha : L\mathcal{D}_8 \otimes L\mathcal{D}_8 \to L\mathcal{D}_8,$$

the spinor representation

$$\beta : L\mathcal{D}_8 \otimes \left(\mathcal{S}^+ \otimes O(K)\right) \to \left(\mathcal{S}^+ \otimes O(K)\right),$$

and the above anti-symmetric homomorphism

$$\gamma : \left(\mathcal{S}^+ \otimes O(K)\right) \otimes \left(\mathcal{S}^+ \otimes O(K)\right) \to L\mathcal{D}_8,$$

to give an alternative way to define the Lie algebra structure on

$$L\mathcal{E}_8 = L\mathcal{D}_8 + \mathcal{S}^+ \otimes O(K).$$

Namely if $a + u$ and $b + v$ are two local section of $L\mathcal{E}_8$ with $a, b \in L\mathcal{D}_8$ and $u, v \in \mathcal{S}^+ \otimes O(K)$, then we define their bracket as

$$\begin{aligned} & [a + u, b + v] \\ = \quad & \alpha(a \otimes b) + \gamma(u \otimes v) + \beta(a \otimes v - b \otimes u). \end{aligned}$$

It is shown in chapter six of [Adams] that fiberwise this algebraic structure is isomorphic the Lie algebra structure of E_8.

To obtain $L\mathcal{E}_n$ for n less than eight, all we need to do is to choose $8-n$ disjoint lines L_j's on X_8 and take away all those line bundle summands of $L\mathcal{E}_8$ which are written as $O(l - L_j)$ or $O(L_j - l)$ for l and L_j disjoint lines, we also take away the summand spanned by homology classes of L_j's in $\Lambda \otimes_{\mathbb{Z}} O_X = O_X^{\oplus 8}$. Then such bundle $L\mathcal{E}_n$ is pullback of a E_n bundle from the blowdown of X_8 along L_j's.

There is another way to obtain $L\mathcal{E}_7$: We first choose two disjoint lines L_1 and L_2 on X_8. We consider the A_1-bundle

$$\mathcal{A}_1 = O(L_1 - L_2) \oplus O \oplus O(L_2 - L_1)$$

on X_8 which is a Lie algebra subbundle of $L\mathcal{D}_8 \subset L\mathcal{E}_8$. Then the centralizer bundle of $\mathcal{A}_1$ inside $L\mathcal{E}_8$ is a E_7-bundle on X_8 which in fact comes from pullback of a E_7 bundle on the blow down of X_8 along the line in the class $H - L_1 - L_2$. However we do not know if a similar method would produce other E_n bundle with $n < 7$.

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