A general Plücker formula

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Abstract

We prove a formula which compares intersection numbers of conormal varieties of two projective varieties and their dual varieties.

When one of them is linear, we can recover the usual Plücker formula for the degree of the dual variety.

The basic strategy of the proof is to study a category of Lagrangian subvarieties in the cotangent bundle of a projective space under a birational transformation.

1 The formula

The conormal variety C_S of any subvariety S in $\mathbb{P}^n$ is a Lagrangian subvariety in $T^*\mathbb{P}^n$ with respect to the canonical holomorphic symplectic form. If S is smooth then its Euler characteristic $\chi(S)$ equals to the intersection number $C_S \cdot \mathbb{P}^n$ up to a sign $(-1)^{\dim S}$. In general $\chi(S)$ is replaced by the Euler characteristic of the Euler obstruction Eu_S defined by MacPherson [Ma], we denote it as $\bar{\chi}(S)$. More generally $C_{S_1} \cdot C_{S_2}$ is well-defined and equals to the Euler characteristic of the intersection up to a sign provided that S_1 and S_2 intersect transversely along a smooth subvariety in $\mathbb{P}^n$.

In this paper we prove the following formula.

Theorem 1 *Suppose S_1 and S_2 are two subvarieties in $\mathbb{P}^n$ which intersect transversely and the same holds true for their dual varieties $S_1^\vee$ and $S_2^\vee$ in $\mathbb{P}^{n*}$. Then we have*

$$C_{S_1} \cdot C_{S_2} + \frac{(C_{S_1} \cdot \mathbb{P}^n)(C_{S_2} \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)} = C_{S_1^\vee} \cdot C_{S_2^\vee} + \frac{\left(C_{S_1^\vee} \cdot \mathbb{P}^{n*}\right)\left(C_{S_2^\vee} \cdot \mathbb{P}^{n*}\right)}{(-1)^{n+1}(n+1)}.$$

This formula arises as we studied in [Le] the Legendre transformation on the category of Lagrangian subvarieties in a hyperkähler manifold under a birational transformation of the ambient manifold.

By applying this formula to cases when S_2 are linear subspaces in $\mathbb{P}^n$ with different dimensions, we obtain the following three corollaries which determine the *degree*, the *Euler characteristic* and the *dimension* of the dual variety. We note that the degree of S_1 equals $C_{S_1} \cdot C_{S_2}$ with S_2 a linear subspace of complementary dimension.

First we recover the generalized Plücker formula of Parusiński [Pa] and Ernström [Er].

Corollary 2 *For any k we have*

$$\deg S^\vee = (-1)^{n-k+1}\left(k\bar{\chi}(S) - (k+1)\bar{\chi}\left(S^1\right) + \bar{\chi}\left(S^{k+1}\right)\right),$$

where S^k is the intersection of S with a generic codimensional k linear subspace.

When S is a plane curve in $\mathbb{P}^2$ this gives the classical Plücker formula for the degree of the dual curve $S^\vee$,

$$\deg S^\vee = d(d-1) - 2\delta - 3\kappa,$$

where d, δ, κ denote the degree, the number of nodes, the number of cusps of S. This classical formula has another generalization for higher dimensional S

with only isolated singularities by Teissier [Te] in the hypersurface case and by Kleiman [Kl] in general. For subjects closely related to the Plücker formula, readers can consult [GKZ] and [Kl2].

Second, we can derive the Euler characteristic of the dual variety.

Corollary 3 *For any subvariety S in $\mathbb{P}^n$ we have*

$$\bar{\chi}\left(S^{\vee}\right) = n\bar{\chi}\left(S\right) - \left(n+1\right)\bar{\chi}\left(S^{1}\right).$$

Third we can determine the dimension of the dual variety by comparing $\chi\left(S^{k}\right)$ with a linear function in k. For example, in the hypersurface case we have,

Corollary 4 *Let S be a hypersurface in $\mathbb{P}^n$, then $S^{\vee}$ has codimension c if and only if for any $k \leq c$ we have*

$$\bar{\chi}\left(S^{k}\right) = k\bar{\chi}\left(S^{1}\right) + \left(1-k\right)\bar{\chi}\left(S\right)$$

and it becomes a strict inequality when $k = c+1$.

There are dimension formulas for dual varieties in terms of Hessian matrices given by Segre [Se] and Katz [Ka].

When S_2 is a smooth quadric hypersurface, its dual variety $S_2^{\vee}$ is again a smooth quadric hypersurface. In this case we have the following corollary.

Corollary 5 *Suppose a subvariety S in $\mathbb{P}^n$ and its dual variety $S^{\vee}$ are both hypersurfaces then*

$$\bar{\chi}\left(S\right) - \hat{\chi}\left(S\cap Q\right)\left(1+\frac{1+\left(-1\right)^{n}}{2n}\right) = \bar{\chi}\left(S^{\vee}\right) - \hat{\chi}\left(S^{\vee}\cap Q'\right)\left(1+\frac{1+\left(-1\right)^{n}}{2n}\right),$$

where Q, Q' are general quadric hypersurfaces and $\hat{\chi}\left(S\cap Q\right) = \left(-1\right)^{n-2} C_S \cdot C_Q$.

Remark: We expect that $\hat{\chi}\left(S\cap Q\right)$ is simply equal to $\bar{\chi}\left(S\cap Q\right)$.

Proof: By applying our formula to the case when S_1 is a hyperplane and $S_2 = Q_{n-1}$ is a general quadric hypersurface in $\mathbb{P}^n$, we obtain

$$\chi\left(Q_n\right) = n+1 \text{ if } n \text{ is odd,}$$
$$\chi\left(Q_n\right) = n+2 \text{ if } n \text{ is even.}$$

This can also be obtained by using explicit descriptions of Q_n's. We apply our formula again by replacing S_1 with the hypersurface S and we obtain the result. ■

2 Proof of the formula

The basic idea of our proof is to study the intersection of two Lagrangian subvarieties C_1 and C_2 in the holomorphic symplectic manifold $T^*\mathbb{P}^n$ and their behaviors under the canonical birational transformation from $T^*\mathbb{P}^n$ to $T^*\mathbb{P}^{n*}$:

Recall that we can write $\mathbb{P}^n = (V\backslash 0)/\mathbb{C}^\times$ and $\mathbb{P}^{n*} = (V^*\backslash 0)/\mathbb{C}^\times$ with V a vector space of dimension $n+1$. There is a similar description for their cotangent bundles, namely

$$T^*\mathbb{P}^n = \{(x,\xi) \in (V\backslash 0) \times V^* : \xi(x) = 0\}/\mathbb{C}^\times,$$
$$T^*\mathbb{P}^{n*} = \{(x,\xi) \in V \times (V^*\backslash 0) : \xi(x) = 0\}/\mathbb{C}^\times.$$

The zero section P in $T^*\mathbb{P}^n$ (resp. P^* in $T^*\mathbb{P}^{n*}$) is given by $\{\xi = 0\}$ (resp. $\{x = 0\}$). The identity homomorphism on $V \times V^*$ descends to a birational map,

$$\Phi : T^*\mathbb{P}^n \dashrightarrow T^*\mathbb{P}^{n*},$$

which is biregular from $T^*\mathbb{P}^n\backslash P$ to $T^*\mathbb{P}^{n*}\backslash P^*$.

Even though $T^*\mathbb{P}^n$ is incomplete, the intersection of C_{S_1} and C_{S_2} only occurs along the zero section P in $T^*\mathbb{P}^n$ when S_1 and S_2 intersect transversely in $\mathbb{P}^n$.[1] This holds true even after we compactify $T^*\mathbb{P}^n$ to M as follow,

$$M = \mathbb{P}(T^*\mathbb{P}^n \oplus O_{\mathbb{P}^n}).$$

If we denote the closure of C_i's in M as $\bar{C}_i$'s, then we have

$$\begin{array}{ccccc} \bar{C}_1 \cap \bar{C}_2 & = & C_1 \cap C_2 & = & S_1 \cap S_2 \\ \cap & & \cap & & \cap \\ M & \supset & T^*\mathbb{P}^n & \supset & \mathbb{P}^n. \end{array}$$

Therefore we can simply write $\bar{C}_1 \cdot \bar{C}_2$ as $C_1 \cdot C_2$. Similarly we have $C_i \cdot P = \bar{C}_i \cdot P$.

Remark: We recall the following useful result (see e.g. [Le]): If C_1 and C_2 are Lagrangian subvarieties in $T^*\mathbb{P}^n$ which intersect cleanly. Suppose their closures $\bar{C}_1$ and $\bar{C}_2$ have the same intersection, i.e. $C_1 \cap C_2 = \bar{C}_1 \cap \bar{C}_2$, then

$$C_1 \cdot C_2 = (-1)^{\dim C_1 \cap C_2} \chi(C_1 \cap C_2).$$

In particular we have $P \cdot P = (-1)^n (n+1)$.

It is not difficult to see that the above birational map Φ on $T^*\mathbb{P}^n$ extends to a birational map between two complete varieties M and M':

$$\Phi_M : M \dashrightarrow M',$$

which is biregular outside P and P^*.

[1] For now on we will simply write C_i for C_{S_i}, the conormal variety of S_i.

Note that M' can also be described as a flop as follow:

$$\begin{array}{ccccc} M & \overset{\pi}{\leftarrow} & \widetilde{M} & \overset{\pi'}{\rightarrow} & M' \\ \cup & & \cup j & & \cup \\ P & \overset{p}{\leftarrow} & \widetilde{P} & \overset{p'}{\rightarrow} & P^*. \end{array}$$

Here $\widetilde{M}$ is the blow up of M along P. The exceptional locus $\widetilde{P} = \pi^{-1}(P)$ is a $\mathbb{P}^{n-1}$-bundle over P. It admits another $\mathbb{P}^{n-1}$-bundle structure over the dual projective space P^* and π' is the blow down of $\widetilde{M}$ along this second fiber structure on $\widetilde{P}$.

In the next section, we will show that the Legendre functor $\mathbf{L} = \pi'_* \circ \pi^{-1}$ on the derived categories of coherent sheaves on M and M' is an equivalence of categories. This implies that

$$Ext^k_{O_M}(S_1, S_2) \cong Ext^k_{O_{M'}}(\mathbf{L}(S_1), \mathbf{L}(S_2)),$$

for any coherent sheaves S_1, S_2 on M. In order to use this equivalence to prove the theorem 1, we need to compute the Chern characters of $\mathbf{L}(O_P)$ and $\mathbf{L}(O_{C_i})$'s. First we relate the Lagrangian intersection number $C_1 \cdot C_2$ with the category of coherent sheaves on M.

Lemma 6 *For any n dimensional subvarieties $\bar{C}_1, \bar{C}_2$ in M we have*

$$\textstyle\sum_k \dim (-1)^k Ext^k_{O_M}\left(O_{\bar{C}_1}, O_{\bar{C}_2}\right) = (-1)^n \bar{C}_1 \cdot \bar{C}_2.$$

Proof: We recall the Riemann-Roch formula for the global Ext groups: For any coherent sheaves S_1 and S_2 on M we have,

$$\textstyle\sum_k \dim (-1)^k Ext^k_{O_M}(S_1, S_2) = \int_M \overline{ch}(S_1)\, ch(S_2)\, Td_M$$

where $\overline{ch}(S_1) = \Sigma(-1)^k ch_k(S_1)$. For $S_i = O_{\bar{C}_i}$ the structure sheaf of a subvariety $\bar{C}_i$ of dimension n, we have

$$\begin{aligned} ch_k\left(O_{\bar{C}_i}\right) &= 0 \text{ for } k < n, \\ ch_n\left(O_{\bar{C}_i}\right) &= \left[\bar{C}_i\right], \end{aligned}$$

where $\left[\bar{C}_i\right]$ denotes the Poincaré dual of the variety $\bar{C}_i$.

Therefore

$$\begin{aligned} &\dim (-1)^k Ext^k_{O_M}\left(O_{\bar{C}_1}, O_{\bar{C}_2}\right) \\ &= \int_M \overline{ch}\left(O_{\bar{C}_1}\right) ch\left(O_{\bar{C}_2}\right) Td_M \\ &= \int_M \left((-1)^n \left[\bar{C}_1\right] + h.o.t.\right)\left(\left[\bar{C}_2\right] + h.o.t.\right)(1 + h.o.t.) \\ &= (-1)^n \int_M \left[\bar{C}_1\right] \cup \left[\bar{C}_2\right] = (-1)^n \bar{C}_1 \cdot \bar{C}_2. \end{aligned}$$

Here *h.o.t.* refers to *higher order terms* which do not contribute to the outcome of the integral. Hence the result. ■

Lemma 7

$$ch\left(\mathbf{L}\left(O_P\right)\right) = \pm\left[P^*\right] + h.o.t..$$

Proof: From the previous lemma, we have

$$\begin{aligned}
&\sum\left(-1\right)^k \dim Ext^k_{O_M}\left(O_P, O_P\right)\\
&= \left(-1\right)^n P\cdot P\\
&= \chi\left(P\right)\\
&= \left(n+1\right).
\end{aligned}$$

On the other hand, the support of $\mathbf{L}\left(O_P\right)$ must be inside $P^* \subset M'$ because M and M' are isomorphic outside their exceptional loci P and P^*. This implies that $ch\left(\mathbf{L}\left(O_P\right)\right) = \alpha\left[P^*\right] + h.o.t.$ for some integer α. Therefore,

$$\begin{aligned}
&\textstyle\sum_k \left(-1\right)^k \dim Ext^k_{O_{M'}}\left(\mathbf{L}\left(O_P\right), \mathbf{L}\left(O_P\right)\right)\\
&= \int_{M'} \overline{ch}\left(\mathbf{L}\left(O_P\right)\right) ch\left(\mathbf{L}\left(O_P\right)\right) Td_M\\
&= \int_{M'} \left(-1\right)^n \alpha\left[P^*\right]\cdot\alpha\left[P^*\right]\\
&= \left(n+1\right)\alpha^2.
\end{aligned}$$

Now $\mathbf{L}$ being an equivalence of categories implies that

$$\sum\left(-1\right)^k \dim Ext^k_{O_{M'}}\left(\mathbf{L}\left(O_P\right), \mathbf{L}\left(O_P\right)\right) = \sum\left(-1\right)^k \dim Ext^k_{O_M}\left(O_P, O_P\right).$$

This forces α to be ± 1. Hence the claim. ■

Lemma 8 *Suppose C is a n dimensional irreducible subvariety in M not containing P. We consider*

$$S = \bigoplus^{n+1} O_C - \left(-1\right)^n \bigoplus^{C\cdot P} O_P.$$

Then

$$ch\left(\mathbf{L}\left(S\right)\right) = \left(n+1\right)\left[C^\vee\right] - \left(-1\right)^n\left(C^\vee\cdot P^*\right)\left[P^*\right] + h.o.t.$$

Proof: First we have

$$\begin{aligned}
&\sum\left(-1\right)^k \dim Ext^k_{O_M}\left(S, O_P\right)\\
&= \left(-1\right)^n\left[\left(n+1\right)C\cdot P - \left(-1\right)^n\left(C\cdot P\right)\left(P\cdot P\right)\right]\\
&= 0.
\end{aligned}$$

Therefore, using the equivalence of categories induced by $\mathbf{L}$ again, we have

$$\sum\left(-1\right)^k \dim Ext^k_{O_{M'}}\left(\mathbf{L}\left(S\right), \mathbf{L}\left(O_P\right)\right) = 0.$$

Away from P^*, the support of $\mathbf{L}(S)$ is clearly $C^\vee$ with multiplicity $n+1$, therefore we have

$$ch(\mathbf{L}(S)) = (n+1)[C^\vee] + \beta[P^*] + h.o.t.$$

for some integer β. On the other hand

$$\begin{aligned}
&\sum (-1)^k \dim Ext^k_{O_{M'}}(\mathbf{L}(S), \mathbf{L}(O_P)) \\
&= \int_{M'} \overline{ch}(\mathbf{L}(S))\, ch(\mathbf{L}(O_P))\, Td_{M'} \\
&= (-1)^n ((n+1)[C^\vee] + \beta[P^*]) \cdot (\pm[P^*]).
\end{aligned}$$

This being zero implies that $\beta = -(-1)^n (C^\vee \cdot P^*)$. Therefore

$$ch(\mathbf{L}(S)) = (n+1)[C^\vee] - (-1)^n (C^\vee \cdot P^*)[P^*] + h.o.t.$$

Hence the result. ■

Now we suppose that C_1 and C_2 are two n dimensional irreducible subvarieties in M which do not include P. We denote the corresponding sheaves constructed as above by S_1 and S_2 respectively. We have

$$\begin{aligned}
&\sum (-1)^k \dim Ext^k_{O_M}(S_1, S_2) \\
&= \int_M \overline{ch}(S_1)\, ch(S_2)\, Td_M \\
&= (n+1)^2 (-1)^n \left(C_1 + \frac{(C_1 \cdot P)}{(-1)^{n+1}(n+1)} P \right) \cdot \left(C_2 + \frac{(C_2 \cdot P)}{(-1)^{n+1}(n+1)} P \right) \\
&= (n+1)^2 (-1)^n \left[C_1 \cdot C_2 + \frac{(-1)^{n+1}}{n+1} (C_1 \cdot P)(C_2 \cdot P) \right].
\end{aligned}$$

By the above computations of the Chern characters of $ch(\mathbf{L}(S_i))$'s we have a similar formula for $\sum (-1)^k \dim Ext^k_{O_{M'}}(\mathbf{L}(S_1), \mathbf{L}(S_2))$, i.e.

$$\begin{aligned}
&\sum (-1)^k \dim Ext^k_{O_{M'}}(\mathbf{L}(S_1), \mathbf{L}(S_2)) \\
&= (n+1)^2 (-1)^n \left[C_1^\vee \cdot C_2^\vee + \frac{(-1)^{n+1}}{n+1} (C_1^\vee \cdot P^*)(C_2^\vee \cdot P^*) \right].
\end{aligned}$$

Finally using the equivalence of categories induced by $\mathbf{L}$ we have

$$\sum (-1)^k \dim Ext^k_{O_M}(S_1, S_2) = \sum (-1)^k \dim Ext^k_{O_{M'}}(\mathbf{L}(S_1), \mathbf{L}(S_2)).$$

Combining these and we obtain

$$C_{S_1} \cdot C_{S_2} + \frac{(C_{S_1} \cdot P)(C_{S_2} \cdot P)}{(-1)^{n+1}(n+1)} = C_{S_1^\vee} \cdot C_{S_2^\vee} + \frac{(C_{S_1^\vee} \cdot P^*)(C_{S_2^\vee} \cdot P^*)}{(-1)^{n+1}(n+1)}.$$

Thus we have completed the proof of the main theorem 1 assuming that $\mathbf{L}$ is an equivlance of categories. ■

3 Equivalence of derived categories

In this section we adapt Bondal and Orlov arguments [BO] to prove that $D^b(M)$ and $D^b(M')$ are equivalent categories. Recall that $\widetilde{M}$ is the blow up of M along $P \cong \mathbb{P}^n$, or the blow up of M' along $P^* \cong \mathbb{P}^{n*}$. We recall the following commutative diagram:

$$\begin{array}{ccccc} M & \overset{\pi}{\leftarrow} & \widetilde{M} & \overset{\pi'}{\rightarrow} & M' \\ \cup & & \cup j & & \cup \\ P & \overset{p}{\leftarrow} & \widetilde{P} & \overset{p'}{\rightarrow} & P^*. \end{array}$$

Proposition 9 *In the above situation, we have an equivalence of derived categories,*

$$\mathbf{L} = \pi_* \pi'^* : D^b(M') \to D^b(M).$$

Note that $\widetilde{P}$ is a $\mathbb{P}^{n-1}$-bundle over the projective space P (or P^*). Therefore $Pic\left(\widetilde{P}\right) = \mathbb{Z} + \mathbb{Z}$. Every line bundle on $\widetilde{P}$ is isomorphic to $p^* O_P(a) \otimes p'^* O_{P^*}(b)$ for some integers a and b, we denote it as $O_{\widetilde{P}}(a, b)$.

We claim that $O_{\widetilde{M}}\left(\widetilde{P}\right)|_{\widetilde{P}}$ is isomorphic to $O_{\widetilde{P}}(-1, -1)$. By symmetry of M and M' we know it must be of the form $O_{\widetilde{P}}(a, a)$ for some a. For the restriction of $O_{\widetilde{M}}\left(\widetilde{P}\right)$ to a fiber of p equals $O_{\mathbb{P}^{n-1}}(-1)$ because $\widetilde{M}$ is obtained from M by blowing up the smooth center P. This implies that $a = -1$.

As a blown up manifold we have

$$\omega_{\widetilde{M}} = \pi^* \omega_M \otimes O_{\widetilde{M}}\left((n-1)\widetilde{P}\right).$$

This implies that

$$\omega_{\widetilde{M}}|_{\widetilde{P}} = O_{\widetilde{P}}(1-n, 1-n).$$

Proof of proposition: The arguments presented here are basically the same as in section 3 of [BO]. For any $A, B \in D^b(M')$ we want to show that

$$\begin{aligned} Hom(A, B) &\cong Hom(\pi_* \pi'^* A, \pi_* \pi'^* B) \\ &\cong Hom(\pi^* \pi_* \pi'^* A, \pi'^* B). \end{aligned}$$

On the other hand $Hom(A, B) \cong Hom(\pi'^* A, \pi'^* A)$ because the functor $\pi'^* : D^b(M') \to D^b\left(\widetilde{M}\right)$ is full and faithful for any blow up morphism. Therefore it is suffices to show that

$$Hom\left(\bar{A}, \pi'^* B\right) = 0,$$

where $\bar{A}$ is defined by the following exact triangle

$$\pi^* \pi_* \pi'^* A \to \pi'^* A \to \bar{A}.$$

Using $\pi^* : D^b(M) \to D^b\left(\widetilde{M}\right)$ being full and faithful we can obtain $\bar{A} \in D(M)^{\perp} \subset D\left(\widetilde{M}\right)$. An earlier result of Orlov showed that

$$D(M)^{\perp} = \left\langle D(P)_{-n+1}, \cdots, D(P)_{-1} \right\rangle$$

as a semiorthogonal decomposition, where $D(P)_{-k}$ is the full subcategories of $D^b\left(\widetilde{M}\right)$ given by the image of $D^b(P)$ under $j_*\left(O_{\widetilde{P}}(-k) \otimes p^*(\bullet)\right)$. Since P isomorphic to $\mathbb{P}^{n-1}$ we have

$$D(M)^{\perp} = \left\langle j_* O_{\widetilde{P}}(a,b) \right\rangle_{\substack{-n+1 \le b \le -1 \\ -1 \le a-b \le n-1}}$$

From this description we know that $j_* O_{\widetilde{P}}(a,b)$ belongs to both $D(M)^{\perp}$ and $D(M')^{\perp}$ when $-n+1 \le a, b \le -1$. In particular

$$Hom\left(\bar{A}, j_* O_{\widetilde{P}}(a,b)\right) = 0 \text{ when } -n+1 \le a, b \le -1.$$

Because of this $\bar{A} \in D(M)^{\perp}$ has to lie inside the subcategory of $D(M)^{\perp}$ generated by those $j_* O_{\widetilde{P}}(a,b)$ with $a \ge 0$. This implies that $\bar{A} \otimes \omega_{\widetilde{M}} \in D(M')^{\perp}$ because of $\omega_{\widetilde{M}}|_{\widetilde{P}} = O_{\widetilde{P}}(1-n, 1-n)$. That is for any $B \in D^b(M')$ we have

$$Hom\left(\pi'^* B, \bar{A} \otimes \omega_{\widetilde{M}}\right) = 0$$

which is equivalent to

$$Hom\left(\bar{A}, \pi'^* B\right) = 0,$$

by the Serre duality. Hence our proposition. ■

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