

Topological Quantum Field Theory for Calabi-Yau threefolds and G_2 -manifolds

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August 19, 2002

1 Introduction

In the past two decades we witness many fruitful interactions between mathematics and physics. One example is the Donaldson-Floer theory for oriented four manifolds. Physical considerations leads to the discovery of the Seiberg-Witten theory which has profound impact to our understandings of four manifolds. Another example is the Mirror Symmetry for Calabi-Yau manifolds. This duality transformation in the string theory leads to many surprising predictions in the enumerative geometry.

String theory in physics studies a ten dimensional space-time $X \times \mathbb{R}^{3,1}$ with X a six dimensional Riemannian manifold with its holonomy group inside $SU(3)$, the so-called *Calabi-Yau threefold*. Certain parts of the Mirror Symmetry conjecture, as studied by Vafa's group, are specific for Calabi-Yau manifolds of complex dimension *three*. They include the Gopakumar-Vafa conjecture on Gromov-Witten invariants of *arbitrary* genus for Calabi-Yau threefolds, the Ooguri-Vafa conjecture on the relationships between knot invariants and enumerations of holomorphic disks and so on. The key reason is they belong to dualities for G_2 -manifolds. G_2 -manifolds can be naturally interpreted as special Octonion manifolds [23]. For any Calabi-Yau threefold X , the seven dimensional manifold $X \times S^1$ is automatically a G_2 -manifold because of the natural inclusion $SU(3) \subset G_2$.

In recent years, there are many studies of G_2 -manifolds in M-theory including works of Archaya, Atiyah, Gukov, Vafa, Witten, Yau, Zaslow and many others (e.g. [1], [5], [13], [2]).

In the studies of the symplectic geometry of a Calabi-Yau threefold X , we consider unitary flat bundles over three dimensional (special) Lagrangian submanifolds L in X . The corresponding theory for a G_2 -manifold M is called the *special \mathbb{H} -Lagrangian geometry* (or *C-geometry* in [19]). where we consider Anti-Self-Dual (abbrev. ASD) bundles over four dimensional coassociative submanifolds, or equivalently *special \mathbb{H} -Lagrangian submanifolds of type II* [23], (abbrev. \mathbb{H} -SLag) C in M .

Counting ASD bundles over a fixed four manifold C is the well-known theory of Donaldson differentiable invariants, $Don(C)$. Similarly, counting unitary flat

bundles over a fixed three manifold L is Floer's Chern-Simons homology theory $HF_{CS}(L)$. When C is a connected sum $C_1 \#_L C_2$ along a homology three sphere, the relative Donaldson invariants $Don(C_i)$'s take values in $HF_{CS}(L)$ and $Don(C)$ can be recovered from individual pieces by a gluing theorem, $Don(C) = \langle Don(C_1), Don(C_2) \rangle_{HF_{CS}(L)}$ (see e.g. [7]). Similarly when L has a handlebody decomposition $L = L_1 \#_\Sigma L_2$, each L_i determines a Lagrangian subspace \mathcal{L}_i in the moduli space $\mathcal{M}^{flat}(\Sigma)$ of unitary flat bundles over the Riemann surface Σ and Atiyah conjectures that we can recover $HF_{CS}(L)$ from the Floer's Lagrangian intersection homology group of \mathcal{L}_1 and \mathcal{L}_2 in $\mathcal{M}^{flat}(\Sigma)$, $HF_{CS}(L) = HF_{Lag}^{\mathcal{M}^{flat}(\Sigma)}(\mathcal{L}_1, \mathcal{L}_2)$. Such algebraic structures in the Donaldson-Floer theory can be formulated as a Topological Quantum Field Theory (abbrev. TQFT), as defined by Segal and Atiyah [3].

In this paper we propose a TQFT by counting ASD bundles over four dimensional \mathbb{H} -SLags C in any closed (almost) G_2 -manifold M , called \mathbb{H} -SLags cycles. They can be identified as zeros of a naturally defined closed one form on the configuration space of topological cycles. We expect to obtain a homology theory $H_C(M)$ by applying the Witten's Morse theory. When M is non-compact with an asymptotically cylindrical end $X \times [0, \infty)$, then the set of boundary data of relative \mathbb{H} -SLag cycles determines a Lagrangian submanifold \mathcal{L}_M in the moduli space $\mathcal{M}^{SLag}(X)$ of special Lagrangian cycles in the Calabi-Yau threefold X .

When we decompose $M = M_1 \#_X M_2$ along an infinite asymptotically cylindrical neck, it is reasonable to expect to have a gluing formula,

$$H_C(M) = HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}).$$

The main technical difficulty in defining this TQFT rigorously is the *compactness* issue for the moduli space of \mathbb{H} -SLags in M . We do not know how to resolve this problem and our homology groups are only defined in the *formal* sense (and physical sense?).

2 G_2 -manifolds and \mathbb{H} -SLag geometry

We first review some basic definitions and properties of G_2 -geometry, see [19] for more details.

Definition 1 *A seven dimensional Riemannian manifold M is called a G_2 -manifold if the holonomy group of its Levi-Civita connection is inside $G_2 \subset SO(7)$.*

The simple Lie group G_2 can be identified as the subgroup of $SO(7)$ consisting of isomorphism $g : \mathbb{R}^7 \rightarrow \mathbb{R}^7$ preserving the linear three form Ω ,

$$\Omega = f^1 f^2 f^3 - f^1 (e^1 e^0 + e^2 e^3) - f^2 (e^2 e^0 + e^3 e^1) - f^3 (e^3 e^0 + e^1 e^2),$$

where $e^0, e^1, e^2, e^3, f^1, f^2, f^3$ is any given orthonormal frame of \mathbb{R}^7 . Such a three form, or up to conjugation by elements in $GL(7, \mathbb{R})$, is called *positive*, and it determines a unique compatible inner product on \mathbb{R}^7 [6].

Gray [12] shows that G_2 -holonomy of M can be characterized by the existence of a positive harmonic three form Ω .

Definition 2 *A seven dimensional manifold M equipped with a positive closed three form Ω is called an almost G_2 -manifold.*

Remark: The relationship between G_2 -manifolds and almost G_2 -manifolds is completely analogous to the relationship between Kahler manifolds and symplectic manifolds.

For example, suppose that X is a complex three dimensional Kähler manifold with a trivial canonical line bundle, i.e. there exists a nonvanishing holomorphic three form Ω_X . Yau's celebrated theorem says that there is a Kähler form ω_X on X with holonomy in $SU(3)$, i.e. a Calabi-Yau threefold. In particular both Ω_X and ω_X are parallel forms. Then the product $M = X \times S^1$ is a G_2 -manifold with

$$\Omega = \text{Re } \Omega_X + \omega_X \wedge d\theta.$$

Conversely, one can prove, using Bochner arguments, every G_2 -metric on $X \times S^1$ must be of this form. More generally, if ω_X is a general Kähler form on X , then $(X \times S^1, \Omega)$ is an *almost* G_2 -manifold and the converse is also true.

Next we quickly review the geometry of \mathbb{H} -SLag cycles in an almost G_2 -manifold (see [19]).

Definition 3 *An orientable four dimensional submanifold C in an almost G_2 -manifold (M, Ω) is called a coassociative submanifold, or simply a \mathbb{H} -SLag, if the restriction of Ω to C is identically zero,*

$$\Omega|_C = 0.$$

If M is a G_2 -manifold, then such a C is calibrated by $*\Omega$ in the sense of Harvey and Lawson [14], in particular, it is an absolute minimal submanifold in M . The normal bundle of any \mathbb{H} -SLag C can be naturally identified with the bundle of self-dual two forms on C . McLean [27] shows that infinitesimal deformations of any \mathbb{H} -SLag are unobstructed and they are parametrized by the space of harmonic self-dual two forms on C , i.e. $H_+^2(C, \mathbb{R})$.

For example, if S is a complex surface in a Calabi-Yau threefold X , then $S \times \{t\}$ is a \mathbb{H} -SLag in $M = X \times S^1$ for any $t \in S^1$. Similarly, if L is a three dimensional special Lagrangian submanifold in X with phase $\pi/2$, i.e. $\omega|_L = \text{Re } \Omega_X|_L = 0$, then $L \times S^1$ is also a \mathbb{H} -SLag in $M = X \times S^1$.

Definition 4 *A \mathbb{H} -SLag cycle in an almost G_2 -manifold (M, Ω) is a pair (C, D_E) with C a \mathbb{H} -SLag in M and D_E an ASD connection over C .*

Remark: \mathbb{H} -SLag cycles are supersymmetric cycles in physics as studied in [26]. Their moduli space admits a natural three form and a cubic tensor [19], which play the roles of correlation function and Yukawa coupling in physics.

We assume that the ASD connection D_E over C has rank one, i.e. a $U(1)$ connection. This avoids the occurrence of reducible connections, thus $\mathcal{M}^{\mathbb{H}\text{-SLag}}(M)$ is a smooth manifold. It has a natural orientation and its expected dimension equals $b^1(C)$, the first Betti number of C . This is because the moduli space of \mathbb{H} -SLags has dimension equals $b_+^2(C)$ [27] and the existence of an ASD $U(1)$ -connection over C is equivalent to $H_-^2(C, \mathbb{R}) \cap H^2(C, \mathbb{Z}) \neq \phi$. The number $b^1(C)$ is responsible for twisting by a flat $U(1)$ -connection.

For simplicity, we assume that $b^1(C) = 0$, otherwise, one can cut down the dimension of $\mathcal{M}^{\mathbb{H}\text{-SLag}}(M)$ to zero by requiring the ASD connections over C to have trivial holonomy around loops $\gamma_1, \dots, \gamma_{b^1(C)}$ in C representing an integral basis of $H_1(C, \mathbb{Z})$. We plan to count the algebraic number of points in this moduli space $\#\mathcal{M}^{\mathbb{H}\text{-SLag}}(M)$.

This number, in the case of $X \times S^1$, can be identified with a proposed invariant of Joyce [17] defined by counting rigid special Lagrangian submanifolds in any Calabi-Yau threefold. To explain this, we need the following proposition on the strong rigidity of product \mathbb{H} -SLags.

Proposition 5 *If $L \times S^1$ is a \mathbb{H} -SLag in $M = X \times S^1$ with X a Calabi-Yau threefold, then any \mathbb{H} -SLag representing the same homology class must also be a product.*

Proof: For simplicity we assume that the volume of the S^1 factor is unity, $Vol(S^1) = 1$. If $L \times S^1$ is a \mathbb{H} -SLag in M then L is special Lagrangian submanifold in X with phase $\pi/2$, i.e. $\text{Re} \Omega_X|_L = \omega|_L = 0$. Suppose C is another \mathbb{H} -SLag in M representing the same homology class, we have $Vol(C) = Vol(L)$. If we write $C_\theta = C \cap (X \times \{\theta\})$ for any $\theta \in S^1$, then $Vol(C_\theta) \geq Vol(L)$, as L is a calibrated submanifold in X . Furthermore the equality sign holds only if C_θ is also calibrated. In general we have

$$Vol(C) \geq \int_{S^1} Vol(C_\theta) d\theta,$$

with the equality sign holds if and only if C is a product with S^1 . Combining these, we have

$$Vol(L) = Vol(C) \geq \int_{S^1} Vol(C_\theta) d\theta \geq \int_{S^1} Vol(L) d\theta = Vol(L).$$

Thus both inequalities are indeed equal. Hence $C = L' \times S^1$ for some special Lagrangian submanifold L' in X . ■

Suppose $M = X \times S^1$ is a product G_2 -manifold and we consider product \mathbb{H} -SLag $C = L \times S^1$ in M . From the above proposition, every \mathbb{H} -SLag representing $[C]$ must also be a product. Since $b_+^2(C) = b^1(L)$, the rigidity of the \mathbb{H} -SLag C in M is equivalent to the rigidity of the special Lagrangian submanifold L

in X . When this happens, i.e. L is a rational homology three sphere, we have $b^2(C) = 0$ and

$$\text{No. of ASD } U(1)\text{-bdl}/C = \#H^2(C, \mathbb{Z}) = \#H^2(L, \mathbb{Z}) = \#H_1(L, \mathbb{Z}).$$

Here we have used the fact that the first cohomology group is always torsion free. Thus the number of such \mathbb{H} -SLag cycles in $X \times S^1$ equals the number of special Lagrangian rational homology three spheres in a Calabi-Yau threefold X , weighted by $\#H_1(L, \mathbb{Z})$. Joyce [17] shows that with this particular weight, the numbers of special Lagrangians in any Calabi-Yau threefold behave well under various surgeries on X , and expects them to be invariants. Thus in this case, we have

$$\#\mathcal{M}^{\mathbb{H}\text{-SLag}}(X \times S^1) = \text{Joyce's proposed invariant for } \#\text{SLag. in } X.$$

In the next section, we will propose a homology theory, whose Euler characteristic gives $\#\mathcal{M}^{\mathbb{H}\text{-SLag}}(M)$.

3 Witten's Morse theory for \mathbb{H} -SLag cycles

We are going to use the parametrized version of \mathbb{H} -SLag cycles in any almost G_2 -manifold M . We fix an oriented smooth four dimensional manifold C and a rank r Hermitian vector bundle E over C . We consider the *configuration space*

$$\mathcal{C} = \text{Map}(C, M) \times \mathcal{A}(E).$$

Definition 6 *An element (f, D_E) in \mathcal{C} is called a parametrized \mathbb{H} -SLag cycles in M if*

$$f^*\Omega = F_E^+ = 0,$$

where the self-duality is defined using the pullback metric from M .

Remark: From the positivity of Ω , $f^*\Omega = 0$ implies f is an immersion.

Instead of $\text{Aut}(E)$, the symmetry group \mathcal{G} in our situation consists of gauge transformations of E which cover *arbitrary* diffeomorphisms on M ,

$$\begin{array}{ccc} E & \xrightarrow{g} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{g_M} & M. \end{array}$$

It fits into the following exact sequence,

$$1 \rightarrow \text{Aut}(E) \rightarrow \mathcal{G} \rightarrow \text{Diff}(C) \rightarrow 1.$$

The natural action of \mathcal{G} on \mathcal{C} is given by

$$g \cdot (f, D_E) = (f \circ g_M, g^*D_E),$$

for any $(f, D_E) \in \mathcal{C} = \text{Map}(C, M) \times \mathcal{A}(E)$. Notice that \mathcal{G} preserves the set of parametrized \mathbb{H} -SLag cycles in M .

The configuration space \mathcal{C} has a natural one form Φ_0 : At any $(f, D_E) \in \mathcal{C}$ we can identify the tangent space of \mathcal{C} as

$$T_{(f, D_E)}\mathcal{C} = \Gamma(C, f^*T_M) \times \Omega^1(C, \text{ad}(E)).$$

We define

$$\Phi_0(f, D_E)(v, B) = \int_C \text{Tr} [f^*(\iota_v \Omega) \wedge F_E + f^* \Omega \wedge B],$$

for any $(v, B) \in T_{(f, D_E)}\mathcal{C}$.

Proposition 7 *The one form Φ_0 on \mathcal{C} is closed and invariant under the action by \mathcal{G} .*

Proof: Recall that there is a universal connection \mathbb{D}_E over $C \times \mathcal{A}(E)$ whose curvature \mathbb{F}_E at a point (x, D_E) equals,

$$\begin{aligned} \mathbb{F}_E|_{(x, D_E)} &= (\mathbb{F}_E^{2,0}, \mathbb{F}_E^{1,1}, \mathbb{F}_E^{0,2}) \\ &\in \Omega^2(C) \otimes \Omega^0(\mathcal{A}) + \Omega^1(C) \otimes \Omega^1(\mathcal{A}) + \Omega^0(C) \otimes \Omega^2(\mathcal{A}) \end{aligned}$$

with

$$\mathbb{F}_E^{2,0} = F_E, \mathbb{F}_E^{1,1}(v, B) = B(v), \mathbb{F}_E^{0,2} = 0,$$

where $v \in T_x C$ and $B \in \Omega^1(C, \text{ad}(E)) = T_{D_E} \mathcal{A}(E)$ (see e.g. [20]). The Bianchi identity implies that $\text{Tr} \mathbb{F}_E$ is a closed form on $C \times \mathcal{A}(E)$. We also consider the evaluation map,

$$\begin{aligned} \text{ev} : C \times \text{Map}(C, M) &\rightarrow M \\ \text{ev}(x, f) &= f(x). \end{aligned}$$

It is not difficult to see that the pushforward of the differential form $\text{ev}^*(\Omega) \wedge \text{Tr} \mathbb{F}_E$ on $C \times \text{Map}(C, M) \times \mathcal{A}(E)$ to $\text{Map}(C, M) \times \mathcal{A}(E)$ equals Φ_0 , i.e.

$$\Phi_0 = \int_C \text{ev}^*(\Omega) \wedge \text{Tr} \mathbb{F}_E.$$

Therefore the closedness of Φ_0 follows from the closedness of Ω . It is also clear from this description of Φ_0 that it is \mathcal{G} -invariant. ■

From now on, we assume that E is a rank one bundle.

Lemma 8 *The zeros of Φ_0 are the same as parametrized \mathbb{H} -SLag cycles in M .*

Proof: Suppose (f, D_E) is a zero of Φ_0 . By evaluating it on various $(0, B)$, we have $f^*\Omega = 0$, i.e. $f : C \rightarrow M$ is a parametrized \mathbb{H} -SLag. This implies that the map

$$\lrcorner\Omega : T_{f(x)}M \rightarrow \Lambda^2 T_x^*C$$

has image equals $\Lambda_+^2 T_x^*C$, for any $x \in C$. By evaluating Φ_0 on various $(v, 0)$, we have $F_E^\perp = 0$, i.e. (f, D_E) is a parametrized \mathbb{H} -SLag cycle in M . The converse is obvious. ■

From above results, Φ_0 descends to a closed one form on \mathcal{C}/\mathcal{G} , called Φ . Locally we can write $\Phi = d\mathcal{F}$ for some function \mathcal{F} whose critical points are precisely (unparametrized) \mathbb{H} -SLag cycles in M . Using the gradient flow lines of \mathcal{F} , we could formally define a Witten's Morse homology group, as in the famous Floer's theory. Roughly speaking one defines a complex (\mathbf{C}_*, ∂) , where \mathbf{C}_* is the free Abelian group generated by critical points of \mathcal{F} and ∂ is defined by counting the number of gradient flow lines between two critical points of relative index one.

Remark: The equations for the gradient flow are given by

$$\frac{\partial f}{\partial t} = *(f^*\xi \wedge F_E), \quad \frac{\partial D_E}{\partial t} = *(f^*\Omega),$$

where $\xi \in \Omega^2(M, T_M)$ is defined by $\langle \xi(u, v), w \rangle = \Omega(u, v, w)$.

The equation

$$\partial^2 = 0$$

requires a good compactification of the moduli space of \mathbb{H} -SLag cycles in M , which we are lacking at this moment. We denote this proposed homology group as $H_C(M)$, or $H_C(M, \alpha)$ when $f_*[C] = \alpha \in H_4(M, \mathbb{Z})$.

This homology group should be invariant under deformations of the almost G_2 -metric on M and its Euler characteristic equals,

$$\chi(H_C(M)) = \#\mathcal{M}^{\mathbb{H}\text{-SLag}}(M).$$

Like Floer homology groups, they measure the *middle dimensional* topology of the configuration space \mathcal{C} divided by \mathcal{G} .

4 TQFT of \mathbb{H} -SLag cycles

In this section we study complete almost G_2 -manifold M_i with asymptotically cylindrical ends and the behavior of $H_C(M)$ when a closed almost G_2 -manifold M decomposes into connected sum of two pieces, each with an asymptotically cylindrical end,

$$M = M_1 \#_X M_2.$$

Nontrivial examples of compact G_2 -manifolds are constructed by Kovalev [18] using such connected sum approach. The boundary manifold X is necessary a

Calabi-Yau threefold. We plan to discuss analytic aspects of M_i 's in a future paper [24].

Each M_i 's will define a Lagrangian subspace \mathcal{L}_{M_i} in the moduli space of special Lagrangian cycles in X . Furthermore we expect to have a gluing formula expressing the above homology group for M in terms of the Floer Lagrangian intersection homology group for the two Lagrangian subspaces \mathcal{L}_{M_1} and \mathcal{L}_{M_2} ,

$$H_C(M) = HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}).$$

These properties can be reformulated to give us a topological quantum field theory. To begin we have the following definition.

Definition 9 *An almost G_2 -manifold M is called cylindrical if $M = X \times \mathbb{R}^1$ and its positive three form respect such product structure, i.e.*

$$\Omega_0 = \text{Re}\Omega_X + \omega_X \wedge dt.$$

A complete almost G_2 -manifold M with one end $X \times [0, \infty)$ is called asymptotically cylindrical if the restriction of its positive three form equals to the above one for large t , up to a possible error of order $O(e^{-t})$. More precisely the positive three form Ω of M restricted to its end equals,

$$\Omega = \Omega_0 + d\zeta$$

for some two form ζ satisfying $|\zeta| + |\nabla\zeta| + |\nabla^2\zeta| + |\nabla^3\zeta| \leq Ce^{-t}$.

Remark: If M is an almost G_2 -manifold with an asymptotically cylindrical end $X \times [0, \infty)$, then (X, ω_X, Ω_X) is a complex threefold with a trivial canonical line bundle, but the Kähler form ω_X might not be Einstein. This is so, i.e. a Calabi-Yau threefold, provided that M is a G_2 -manifold. We will simply write $\partial M = X$.

We consider \mathbb{H} -SLags C in M which satisfy a *Neumann condition* at infinity. That is, away from some compact set in M , the immersion $f : C \rightarrow M$ can be written as

$$f : L \times [0, \infty) \rightarrow X \times [0, \infty)$$

with $\partial f / \partial t$ vanishes at infinite [24]. A relative \mathbb{H} -SLag itself has asymptotically cylindrical end $L \times [0, \infty)$ with L a special Lagrangian submanifold in X . A *relative \mathbb{H} -SLag cycle* in M is a pair (C, D_E) with C a relative \mathbb{H} -SLag in M and D_E a unitary connection over C with finite energy,

$$\int_C |F_E|^2 dv < \infty.$$

Any finite energy connection D_E on C induces a unitary flat connection $D_{E'}$ on L [7].

Such a pair $(L, D_{E'})$ of a unitary flat connection $D_{E'}$ over a special Lagrangian submanifold L in a Calabi-Yau threefold X is called a *special Lagrangian cycle* in X . Their moduli space $\mathcal{M}^{SLag}(X)$ plays an important role in

the Strominger-Yau-Zaslow Mirror Conjecture [28] or [22]. The tangent space to $\mathcal{M}^{SLag}(X)$ is naturally identified with $H^2(L, \mathbb{R}) \times H^1(L, ad(E'))$. For line bundles over L , the cup product

$$\cup : H^2(L, \mathbb{R}) \times H^1(L, \mathbb{R}) \rightarrow \mathbb{R},$$

induces a symplectic structure on $\mathcal{M}^{SLag}(X)$ [15]. Using analytic results from [24] about asymptotically cylindrical manifolds, we can prove the following theorem.

Claim 10 *Suppose M is an asymptotically cylindrical (almost) G_2 -manifold with $\partial M = X$. Let $\mathcal{M}^{\mathbb{H}\text{-}SLag}(M)$ be the moduli space of rank one relative \mathbb{H} -SLag cycles in M . Then the map defined by the boundary values,*

$$b : \mathcal{M}^{\mathbb{H}\text{-}SLag}(M) \rightarrow \mathcal{M}^{SLag}(X),$$

is a Lagrangian immersion.

Sketch of the proof ([24]): For any closed Calabi-Yau threefold X (resp. G_2 -manifold M), the moduli space of rank one special Lagrangian submanifolds L (resp. \mathbb{H} -SLags C) is smooth [27] and has dimension $b^2(L)$ (resp. $b_+^2(C)$). The same holds true for complete manifold M with a asymptotically cylindrical end $X \times [0, \infty)$, where $b_+^2(C)_{L^2}$ denote the dimension of L^2 -harmonic self-dual two forms on a relative \mathbb{H} -SLag C in M .

The linearization of the boundary value map $\mathcal{M}^{\mathbb{H}\text{-}SLag}(M) \rightarrow \mathcal{M}^{SLag}(X)$ is given by $H_+^2(C)_{L^2} \xrightarrow{\alpha} H^2(L)$. Similar for the connection part, where the boundary value map is given by $H^1(C)_{L^2} \xrightarrow{\beta} H^1(L)$. We consider the following diagram where (i) each row is a long exact sequence of L^2 -cohomology groups for the pair (C, L) and (ii) each column in a perfect pairing.

$$\begin{array}{ccccccccccc} 0 & \rightarrow & H_+^2(C, L) & \rightarrow & H_+^2(C) & \xrightarrow{\alpha} & H^2(L) & \rightarrow & H^3(C, L) & \rightarrow & \dots \\ & & \otimes & & \otimes & & \otimes & & \otimes & & \\ 0 & \leftarrow & H_+^2(C) & \leftarrow & H_+^2(C, L) & \leftarrow & H^1(L) & \xleftarrow{\beta} & H^1(C) & \leftarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \mathbb{R} & & \end{array}$$

Notice that $H_+^2(C, L)$, $H_+^2(C)$ and $H^2(L)$ parametrize infinitesimal deformation of C with fixed ∂C , deformation of C alone and deformation of L respectively.

By simply homological algebra, it is not difficult to see that $\text{Im } \alpha \oplus \text{Im } \beta$ is a Lagrangian subspace of $H^2(L) \oplus H^1(L)$ with the canonical symplectic structure. Hence the result. ■

We denote the immersed Lagrangian submanifold $b(\mathcal{M}^{\mathbb{H}\text{-}SLag}(M))$ in $\mathcal{M}^{SLag}(X)$ by \mathcal{L}_M . When M decompose as a connected sum $M_1 \#_X M_2$ along a long neck,

as in Atiyah's conjecture on Floer Chern-Simons homology group [3], we expect to have an isomorphism,

$$H_C(M) \cong HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2}).$$

More precisely, suppose Ω_t with $t \in [0, \infty)$, is a family of G_2 -structure on $M_t = M$ such that as t goes to infinite, M decomposes into two components M_1 and M_2 , each has an asymptotically cylindrical end $X \times [0, \infty)$. Then we expect that $\lim_{t \rightarrow \infty} H_C(M_t) \cong HF_{Lag}^{\mathcal{M}^{SLag}(X)}(\mathcal{L}_{M_1}, \mathcal{L}_{M_2})$. We summarize these structures in the following table:

Manifold:	(almost) G_2 -manifold, M^7	(almost) CY threefold, X^6
SUSY Cycles:	\mathbb{H} -SLag. submfs.+ ASD bundles	SLag submfs.+ flat bundles
Invariant:	Homology group, $H_C(M)$	Fukaya category, $Fuk(\mathcal{M}^{SLag}(X))$.

These associations can be formalized to form a TQFT [4]. Namely we associate an additive category $F(X) = Fuk(\mathcal{M}^{SLag}(X))$ to a closed almost Calabi-Yau threefold X , a functor $F(M) : F(X_0) \rightarrow F(X_1)$ to an almost G_2 -manifold M with asymptotically cylindrical ends $X_1 - X_0 = X_1 \cup \bar{X}_0$. They satisfy

- (i) $F(\phi) =$ the additive tensor category of vector spaces $((Vec))$,
- (ii) $F(X_1 \amalg X_2) = F(X_1) \otimes F(X_2)$.

For example, when M is a closed G_2 -manifold, that is a cobordism between empty manifolds, then we have $F(M) : ((Vec)) \rightarrow ((Vec))$ and the image of the trivial bundle is our homology group $H_C(M)$.

5 More TQFTs

There are other TQFTs naturally associated to Calabi-Yau threefolds and G_2 -manifolds but (1) they do not involve nontrivial coupling between submanifolds and bundles and (2) new difficulties arise because of corresponding moduli spaces for Calabi-Yau threefolds have virtual dimension zero and could be singular. They are essentially in the paper by Donaldson and Thomas [9].

TQFT of associative cycles

We assume that M is a G_2 -manifold, i.e. Ω is parallel rather than closed. Three dimensional submanifolds A in M calibrated by Ω is called *associative submanifolds* and they can be characterized by $\chi|_A = 0$ ([14]) where $\chi \in \Omega^3(M, T_M)$ is defined by $\langle w, \chi(x, y, z) \rangle = *\Omega(w, x, y, z)$. We define a *parametrized A-cycle* to be a pair $(f, D_E) \in \mathcal{C}_A = Map(A, M) \times \mathcal{A}(E)$, with

$f : A \rightarrow M$ a parametrized A-submanifold and D_E is a unitary flat connection on a Hermitian vector bundle E over A . There is also a natural \mathcal{G} -invariant closed one form Φ_A on \mathcal{C}_A given by

$$\Phi_A(f, D_E)(v, B) = \int_A \text{Tr} F_E \wedge B + \langle f^* \chi, v \rangle_{T_M},$$

for any $(v, B) \in \Gamma(A, f^* T_M) \times \Omega^1(A, ad(E)) = T_{(f, D_E)} \mathcal{C}_A$. Its zero set is the moduli space of A-cycles in M . As before, we could formally apply arguments in Witten's Morse theory to Φ_A and define a homology group $H_A(M)$.

The corresponding category associated to a Calabi-Yau threefold X would be the Fukaya-Floer category of the moduli space of unitary flat bundles over holomorphic curves in X , denote $\mathcal{M}^{curve}(X)$. We summarize these in the following table:

Manifold:	G_2 -manifold, M^7	CY threefold, X^6
SUSY Cycles:	A-submfds.+ flat bundles	Holomorphic curves+ flat bundles
Invariant:	Homology group, $H_A(M)$	Fukaya category, $Fuk(\mathcal{M}^{curve}(X))$.

TQFT of Donaldson-Thomas bundles

We assume that M is a seven manifold with a G_2 -structure such that its positive three form Ω is co-closed, rather than closed, i.e. $d\Theta = 0$ with $\Theta = *\Omega$. In [9] Donaldson and Thomas introduce a first order Yang-Mills equation for G_2 -manifolds,

$$F_E \wedge \Theta = 0.$$

Their solutions are the zeros of the following gauge invariant one form Φ_{DT} on $\mathcal{A}(E)$,

$$\Phi_{DT}(D_E)(B) = \int_M \text{Tr} [F_E \wedge B] \wedge \Theta,$$

for any $B \in \Omega^1(M, ad(E)) = T_{D_E} \mathcal{A}(E)$. This one form Φ_{DT} is closed because of $d\Theta = 0$. As before, we can formally define a homology group $H_{DT}(M)$. The corresponding category associated to a Calabi-Yau threefold X should be the Fukaya-Floer category of the moduli space of Hermitian Yang-Mills connections over X , denote $\mathcal{M}^{curve}(X)$. Again we summarize these in a table:

Manifold:	G_2 -manifold, M^7	CY threefold, X^6
SUSY Cycles:	DT-bundles	Hermitian YM-bundles
Invariant:	Homology group, $H_{DT}(M)$	Fukaya category, $Fuk(\mathcal{M}^{HYM}(X))$.

It is an interesting problem to understand the transformations of these TQFTs under dualities in M-theory.

Acknowledgments: This paper is partially supported by NSF/DMS-0103355. The author expresses his gratitude to J.H. Lee, R. Thomas, A. Voronov and X.W. Wang for useful discussions.

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