Marta Lewicka, Math 5535, Winter 2005

Homework 6

1. Let $X = \mathbf{R}^2$. Check if (X, d) is a metric space, where the distance $d : X \times X \longrightarrow \mathbf{R}_+$ is defined by:

$$\text{(i)}\ \ d(x,y) = \begin{cases} ((x_1-y_1)^2 + (x_2-y_2)^2)^{1/2} & \text{if the line passing through} \\ & \text{points x and y contains 0,} \\ (x_1^2+x_2^2)^{1/2} + (y_1^2+y_2^2)^{1/2} & \text{otherwise.} \end{cases}$$

$$\text{(ii)}\ \ d(x,y) = \begin{cases} |x_2 - y_2| & \text{if } x_1 = y_1, \\ |x_2| + |y_2| + |x_1 - y_1| & \text{otherwise.} \end{cases}$$

for every $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbf{R}^2$.
In both cases, draw the balls $B_1((0,0))$, $B_1((1,1))$, $B_2((1,1))$.

2. Let X be any set. For each $x, y \in X$ define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Prove that (X, d) is a metric space. d is called the discrete metric.

3. Let (X, d) and (Y, ρ) be two metric spaces and let $f : X \longrightarrow Y$. Recall that we call f continuous iff

$$\forall x \in X\ \forall \epsilon > 0\ \exists \delta > 0\ \forall y \in X\ \ d(x,y) < \delta \implies \rho(f(x), f(y)) < \epsilon.$$

In the following examples check whether the give function is continuous:

(i) any function $f : (X, d) \longrightarrow (Y, \rho)$ where d is the discrete metric (see problem 2) and ρ is any metric,

(ii) the identity function $Id : (\mathbf{R}^2, d) \longrightarrow (\mathbf{R}^2, \rho)$, $Id(x) = x$, where d is one of the metrics in problem 1 and ρ is the Euclidean metric.

(iii) the same as in (ii), but d being the Euclidean metric and ρ one of the metrics defined in problem 1.

4. Let (X, d) and (X, d_1) be two metric spaces. We say that metrics d and d_1 are equivalent if both identity functions $Id_1 : (X, d) \longrightarrow (X, d_1)$ and $Id_2 : (X, d_1) \longrightarrow (X, d)$ are continuous. In each example below, show that (X, d_1) is a metric space and check if d and d_1 are equivalent.

(i) $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$,

(ii) $d_1(x,y) = \min(1, d(x,y))$.

5. In a metric space:

(i) Can an open ball of radius 4 be a proper subset of some open ball of radius 3?

(ii) Show that if an open ball of radius 7 is a subset of some open ball of radius 3, then these two balls must be equal.