MIDTERM 1, Dynamical Systems and Chaos, Winter 2005

Solve problem 1, and any two of the problems 2, 3 and 4. Clearly indicate your choice. Good luck!

Problem 1. (30 points: 5 points each, 1 point for correct answer, 4 points for correct explanation)
Let $f : I \longrightarrow I$ where I is a subinterval (bounded or unbounded) of $\mathbf{R}$. True or False?

- (i) If f is continuous and has a prime 28-cycle then it must also have a prime 21-cycle.
- (ii) If I is a closed interval then f must have at least one fixed point.
- (iii) The orbit of eventual cycle of f may contain infinitely many distinct points.
- (iv) If f is strictly decreasing then it cannot have a cycle of prime period 2.
- (v) The function $f(x) = (586x^2 - \pi)^2 + 568$ has Schwarzian derivative strictly negative at each $x > 1$.
- (vi) Every asymptotically stable fixed point is necessarily attracting.

Problem 2. (35 points = 10 for (i) + 25 for (ii))
Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a $\mathcal{C}^1$ function and let p be its hyperbolic fixed point.

- (i) State the definitions of a hyperbolic fixed point and of an attracting fixed point.
- (ii) Prove (that is, do not just state the result from class) that if $0 < f'(p) \leq 1$ then p is attracting.

Problem 3. (35 points = 10 for (i) + 20 for (ii) + 5 for (iii))

- (i) State the Sharkovsky theorem.
- (ii) Assuming that the existence of a prime 3-cycle implies the existence of a prime cycle of any length, prove that:
  the existence of a prime $3 \cdot 2^m$ cycle implies the existence of a **prime** $5 \cdot 2^{m+1}$ cycle, for every $m \geq 0$.
- (iii) Give an example of a discontinuous function which has a 2-cycle but does not have any fixed point.

Problem 4. (35 points = 10 for (i) + 15 for (ii) + 10 for (iii))
Let p be an asymptotically stable fixed point of a continuous function $f : \mathbf{R} \longrightarrow \mathbf{R}$. Let I be the maximal interval containing p and contained in the stable set $\mathcal{W}^s(p)$.

- (i) Prove that I is open.
- (ii) Prove that I is invariant, that is $f(I) \subset I$.
- (iii) Give an example of f and p so that $\mathcal{W}^s(p) \neq I$.