MIDTERM 2, Dynamical Systems and Chaos, Winter 2005

In problems 1 - 9 determine whether each of the four statements is True or False. You do not have to explain your answer.
In problem 10 write down the four definitions.

Each correct answer is worth 0.25 points. Moreover, each problem solved correctly (all four answers correct) is worth an additional 1 point. The total score for the test is therefore maximum 20 points.

Good luck!

NAME: *SOLUTIONS*

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Problem 1. Let (X, d) be a metric space and let $A \subset X$.

(i) If A is closed then every point of A is its limit point.
FALSE. $A := [0,1] \cup \{2\} \subset R$. *Then* 2 *is not a limit point.*

(ii) The union of any number of closed sets is closed.
FALSE. An open interval $(0,1) = \cup_i [1/i, 1 - 1/i] \subset R^2$.

(iii) If d is the discrete metric then the intersection of any number of open sets is open.
TRUE. Every set is open when d *is the discrete metric.*

(iv) If $\bar{A}$ is dense in X then A is also dense in X.
TRUE. Since $\bar{A}$ *is a closed set, then its density implies* $\bar{A} = X$, *which is the definition of* A *being dense in* X.

Problem 2. Let X, Y be two metric spaces and let $f : X \longrightarrow Y$.

(i) Assume that f is continuous. For some open set $U \subset Y$, the set $f^{-1}(U)$ may be closed in X.
TRUE. Take $U = Y$, *and* X *is of course closed in* X.

(ii) If f is continuous then for every $A \subset X$ there holds: $f(\bar{A}) = \overline{f(A)}$.
FALSE. Take $f(x) = arctg(x)$ *and* $A = R$, *then* $f(A) = (-\pi/2, \pi/2)$ *which is not a closed set in* R.

(iii) If for every $A \subset X$ there holds: $f(\bar{A}) = \overline{f(A)}$, then f must be continuous.
TRUE. This was a homework from the textbook.

(iv) If for every open set $U \subset X$ the set $f(U)$ is open in Y, then f must be continuous.
FALSE. Take $f : R \longrightarrow [-1,1]$ *given by* $f(x) = \sin(1/x)$ *for* $x \neq 0$ *and* $f(0) = 0$. *It is not continuous but it takes open sets into open subsets of* $[0,1]$.

Problem 3.

(i) In $\mathbf{R}$, every two bounded metrics must be equivalent.
FALSE. The discrete metric and the metric $d(x,y) = \max\{|x-y|, 1\}$ *are not equivalent, because the latter one is equivalent to the Euclidean metric.*

(ii) In a metric space, a ball of radius $3\frac{1}{2}$ may be a proper subset of a ball of radius 2.
TRUE. Let $X = \{-1.75, -1, 0, 1, 1.75\}$ *with the Euclidean metric. Then* $B_{3.5}(-1.75) = X \setminus \{1.75\}$ *is a proper subset of* $B_2(0) = X$.

(iii) If d and d_1 are two metrics on X, then their product $d \cdot d_1$ is also a metric.
FALSE. $d(x,y) = |x-y|^2$ *is not a metric on* R *as it does not satisfy the triangle inequality for* $x = -1, y = 0, z = 1$.

(iv) Assume that the identity $\mathrm{Id} : (X, d) \longrightarrow (X, \rho)$ is continuous. If a function $f : (X, \rho) \longrightarrow (X, \rho)$ is continuous then $f : (X, d) \longrightarrow (X, d)$ is also continuous.
FALSE. Let $X = R^2$ *and* d *be the 'river' metric and* ρ *the Euclidean metric. Then* $f(x,y) = (x, y+1)$ *is of course continuous with respect to* ρ *but is discontinuous at* $(0,0)$ *for* d. *This because, for example, the sequence* $(1/n, 0)$ *converges to* $(0,0)$ *but the sequence* $(1/n, 1)$ *does not converge to* $(0,1)$ *in* (X, d).

Problem 4.

(i) Every continuous function $f : [0,1] \longrightarrow \mathbf{R}$ is Lipschitz continuous.
FALSE. $f(x) = \sqrt{x}$ is continuous but not Lipschitz.

(ii) Every periodic, $\mathcal{C}^1$ function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is Lipschitz continuous.
TRUE. The derivative of f must be bounded (since it is continuous and periodic, it is enough to use the fact that a continuous function on a compact set is bounded). Therefore f is Lipschitz continuous, by mean value theorem.

(iii) Let h be a homeomorphism between two metric spaces. If h is Lipschitz continuous then so is h^{-1}.
FALSE. arctg is a homeomorphism between R and $(-\pi/2, \pi/2)$ which is Lipschitz continuous but its inverse tg is not Lipschitz continuous.

(iv) Let $h : (X, d) \longrightarrow (Y, \rho)$ be a homeomorphism. If the metrics d and d_1 are equivalent and if the metrics ρ and ρ_1 are equivalent, then $h : (X, d_1) \longrightarrow (Y, \rho_1)$ is also a homeomorphism.
TRUE. The invertibility of h is clear. Both h and h^{-1} are continuous with respect to metrics d_1 and ρ_1 as appropriate compositions of continuous identities and the continuous maps h and h^{-1} with respect to d and ρ.

Problem 5.

(i) Every continuous map $f : \Sigma_2^+ \longrightarrow \Sigma_2^+$ has a fixed point.
FALSE. Take $f(x) = \{1 - x_0, x_0, x_1, x_2 \ldots\}$ when $x = \{x_0, x_1, x_2, \ldots\}$. This function is even Lipschitz continuous but does not have a fixed point.

(ii) There is a bijection between Σ_2^+ and $[0,1]$.
TRUE. For $x \in \Sigma_2^+$ define $f(x) \in [0,1]$ to be the number with a binary representation $(0.x)_2$. Then f is a bijection.

(iii) The set of all elements $x = \{x_i\}_{i=0}^{\infty}$ of Σ_2^+ which have $x_{10} = 1$ is open.
TRUE. If $x_{10} = 1$ and $d(x, y) < 1/2^{10}$ then also $y_{10} = 1$, by of the homework problems.

(iv) The set of all elements x of Σ_2^+ that have $x_i = 1$ for infinitely many $i : 0 \ldots \infty$, is not closed.
TRUE. The set A of elements x of Σ_2^+ that have $x_i = 1$ for only finitely many $i : 0 \ldots \infty$, is certainly not open. For example $x = (0, 0, 0, \ldots) \in A$, but each open neighbourhood of x contains a point not from A, namely the point $y = \{y_i\}$ with $y_i = 0$ for $i < n$ and $y_i = 1$ for $i \geq n$ where n is sufficiently large.

Problem 6.

(i) Σ_2^+ is a Cantor set.
TRUE. Proven in class; we constructed a homeomorphism between Σ_2^+ *and* Λ *which is a Cantor set.*

(ii) A homeomorphic image of a Cantor set is a Cantor set.
TRUE. Check that all the properties still hold.

(iii) Let C_1 and C_2 be two subsets of $\mathbf{R}$. If C_1 and C_2 are Cantor sets (with the Euclidean metric) then so is $C_1 \cup C_2$.
TRUE. Check all the parts of the definition: compact, perfect and completely disconnected.

(iv) In the ternary Cantor set, there is a point with a ternary representation containing the digit 1.
TRUE. $1/3 = (0.1)_3$ *belongs to the ternary Cantor set, because it has another representation* $1/3 = (0.2222222....)_3$.

Problem 7.

(i) There exists a continuous function which is transitive but the set of its periodic points is not dense in its domain.
TRUE. Rotation by an irrational angle - done in class.

(ii) For every continuous function $f : \mathbf{R} \longrightarrow \mathbf{R}$ sensitive dependence on initial conditions implies transitivity.
FALSE. $f(x) = 2x$ *has sensitive dependence on initial conditions (shown in class) but is not transitive.*

(iii) If $f : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous and has a dense orbit then it must have sensitive dependence on initial conditions.
TRUE. Continuity and transitivity on R implies chaos (shown in class) and therefore it also implies sensitive dependence on initial conditions.

(iv) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be continuous and such that for every two closed intervals A, B, there exists n such that $f^{(n)}(A) \supset B$. Then f must be chaotic.
TRUE. It is enough to notice that f *must be transitive. Indeed, if* U, V *are two nonempty open sets, then they must contain some closed (nonempty) intervals* A, B *and we have:* $f^{(n)}(U) \cap V \supset f^{(n)}(A) \cap B = B \neq \emptyset$.

Problem 8.

(i) The logistic map F_4 is chaotic on $[0, 1]$.
TRUE. F_4 *is conjugate to the tent map on* $[0, 1]$*, which is chaotic by homework.*

(ii) The logistic map F_1 is chaotic on $[0, 1]$.
FALSE. $F_1([0, 1]) = [0, 1/4]$ *so it cannot be transitive.*

(iii) Every rotation map on S^1 which is not the identity, must be transitive.
FALSE. A rotation by π *is not transitive.*

(iv) The map $f : S^1 \longrightarrow S^1$ given by: $f(e^{i\theta}) = e^{-3i\theta}$ has sensitive dependence on initial conditions.
TRUE. Proven exactly as the same property of $f(e^{i\theta}) = e^{2i\theta}$ *which was discussed in class.*

Problem 9. Let $A, B \in M^{2\times 2}$.

(i) If A is similar to B then the linear functions $f, g : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ given as: $f(x) = Ax$ and $g(x) = Bx$ are conjugate.
TRUE. The homeomorphism is given by multiplication by the invertible similarity matrix P such that $AP = PB$.

(ii) If $\det A = 0$ then A must be similar to $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, for some $\lambda_1,\ \lambda_2$.
FALSE. The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a counterexample.

(iii) The matrix $\begin{bmatrix} 20 & 2 \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$.
FALSE. The first matrix has 2 eigenvalues 20 and 0 while the second rotation matrix does not have any (real) eigenvalue. If the matrices were similar then they would have the same eigenvalues.

(iv) If A has only one eigenvalue then for every $n \geq 1$, A^n has only one eigenvalue.
TRUE. Use the characterisation theorem.

Problem 10. Write (precisely) the definitions of:

(i) A Cantor set.
(ii) Sensitive dependence on initial conditions.
(iii) Transitivity.
(iv) Conjugacy.
Check the textbook or your class notes.