MIDTERM 3, Dynamical Systems and Chaos, Winter 2005

Solve problem 1, and any two of the problems 2, 3 and 4. Clearly indicate your choice. Good luck!

Problem 1. (30 points: 5 points each, 1 point for correct answer, 4 points for correct explanation)
Let $A, B \in M^{2\times 2}$. True or False?

(i) If A and B have the same eigenvalues $\lambda_1 \neq \lambda_2$ then A must be similar to B.

(ii) $A = 2 \cdot Id$ induces a hyperbolic toral automorphism.

(iii) If a fixed point x of a continuous map $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ is attracting then it must be isolated, that is: a sufficiently small neighbourhood of x does not contain any other fixed point of f but x.

(iv) If A^2 has two distinct eigenvalues then the same is true for A.

(v) 0 is an asymptotically stable fixed point of the linear map $f(x) = Ax$ with $A = \begin{bmatrix} -6 & 1/2 \\ -1 & 0 \end{bmatrix}$.

(vi) There exists a (nonzero) sequence $\{x_n\}$ satisfying: $x_{n+2} + 6x_{n+1} + 9x_n = 0$ and $\lim_{n\to\infty} |x_{n+1}/x_n| = \infty$.

Problem 2. (35 points = 10 for (i) + 10 for (ii) + 10 for (iii) + 5 for (iv))
Let $A \in M^{2\times 2}$.

(i) Prove (that is, do not just state a theorem from class) that if A has two distinct eigenvalues then the corresponding eigenvectors are linearly independent.

(ii) Prove that A as in (i) must be similar to a diagonal matrix.

(iii) Find the general solution of the equation:

$$X(n+1) = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} X(n).$$

(iv) Draw the phase-space diagram in the vicinity of the origin, for the following discrete dynamical system:

$$\begin{aligned} x_{n+1} &= 2\sin y_n \\ y_{n+1} &= 4(x_n - y_n) - 4\sin x_n - 2e^{y_n} + 2 \end{aligned}$$

Problem 3. (35 points = 5 for (i) + 5 for (ii) + 15 for (iii) + 10 for (iv))

(i) Let $(0, 0)$ be a fixed point of a continuous map $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$. State (in detail) the definition of the asymptotic stability of 0.

(ii) Is $(0, 0)$ an asymptotically stable fixed point of the following function?:

$$f(x, y) = (y - 2xy, x/2 + xy^2)$$

(iii) Find a Lyapunov function for f as in (ii), in the vicinity of the origin.

(iv) Draw the phase-space diagram in the vicinity of the origin, for the iterations of the function f given in (ii).

Problem 4. (35 points = 20 for (i) + 5 for (ii) + 10 for (iii))
Consider the map $F : S \longrightarrow \mathbf{R}^2$, defined on the unit square S, as in the picture below. F contracts vertical lengths and expands horisontal lengths of S as in the case of the Smale horseshoe, but bends S in a different way:

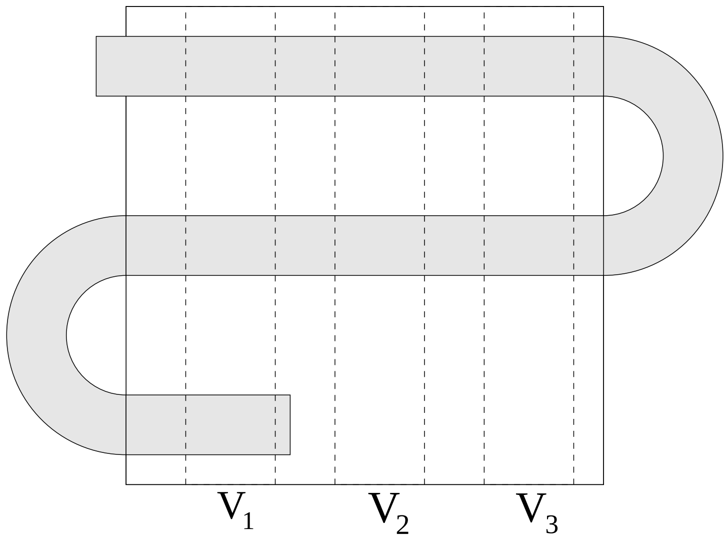

FIGURE 1

Define $\Lambda := \{x \in S;\ F^{(n)}(x) \in S\ \forall n \in \mathbf{N}\}$.

(i) Show that Λ is homeomorphic to the space $\Sigma_3^+(A)$, for some transition matrix $A \in M^{3\times 3}$. What is A?
(ii) Prove that $F(\Lambda) \subset \Lambda$.
(iii) Prove that $F : \Lambda \longrightarrow \Lambda$ is conjugate to the shift map σ on $\Sigma_3^+(A)$.