A note on separation theorems

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Abstract

The first part of this note includes the separations theorems [1]. We noticed that for two convex sets to be separable, it's essential that one of the sets has nonempty interior. In the second part, examples are given to illustrate the point[2].

1 Separation theorems

Theorem 1. *(First Separation Theorem)*
Let X be a Banach space, $E \subset X$ an nonempty open convex set in E, and $x \in X \setminus E$. Then there exists a continuous linear functional T and $\alpha \in \mathbb{R}$ such that

$$T(E) < \alpha \leqslant T(x)$$

Remark 2. *Proof follows from Hahn-Banach Theorem applied on a Minkovski functional determined by the open convex set E. Hahn-Banach Theorem only requires a locally convex topological vector space. So the separation theorem is also valid there.*

Corollary 3. *Let X be a Banach space, A, B are two disjoint nonempty convex subsets of X, and A is open. Then there exists a continuous linear functional T and $\alpha \in \mathbb{R}$ such that*

$$T(A) < \alpha \leqslant T(B).$$

Proof. A is open, so $A - B$ is open and $0 \notin A - B$. Apply the theorem. □

Corollary 4. *(Second Separation Theorem) Let X be a Banach space, A, B are two disjoint nonempty convex subsets of X, A is compact and B is closed. Then there exists a continuous linear functional T and $\alpha \in \mathbb{R}$ such that*

$$T(A) < \alpha < \beta < T(B)$$

Proof. There exists $B(0, \epsilon)$ a ball of radius ϵ centered at 0 such that $A+B(0, \epsilon)$ open, convex and $(A + B(0, \epsilon)) \cap B = \varnothing$. Apply the previous corollary. □

Corollary 5. *Let X be a Banach space, A, B are two disjoint nonempty convex subsets of X, A has nonempty interior. Then there exists a continuous linear functional T and $\alpha \in \mathbb{R}$ such that*

$$T(A) \leqslant \alpha \leqslant T(B)$$

Proof. Shrink the set A to be an open set. Apply the previous corollary. Then $T(\partial A) \leqslant \alpha$ because of the continuity of T and the definition of boundary of a set. □

2 Examples of nonseparable closed sets

Remark 6. *Notice that in the previous forms of separation theorems, it is essential that one of the two disjoint sets has interior point. Then things are reduced to the First Separation Theorem. If this condition is not satisfied, 2 sets may not be separated by a continuous linear functional.*

Example 7. *In the normed space l^1 of summable sequence of real numbers $x = (x_0, x_1, x_2, \cdots,)$, let D be the line defined by the relations $x_n = 0$ for $n > 0$. And A is defined by $x_0 \geqslant |\alpha_n x_n - \beta_n|$, where $\alpha_n = n^3$ and $\beta_n = n$. Then we can show A is closed, convex, $A \cap D = \varnothing$ and that A, D can not be separated. (See [2] page II.78, ex 10)*

Proof. A is the intersection of closed sets $E_n = \{x \in l^1 | \; x_0 \geqslant |\alpha_n x_n - \beta_n|\}$, so A is closed. It is easy to verify that A is convex. Moreover, if $(x_0, 0, 0, \cdots) \in A$, then $x_0 \geqslant |\beta_n|$. That is, $x_0 = \infty$. So $A \cap D = \varnothing$.

$A - D$ is dense in the whole space l^1. Let $x = (x_0, x_1, x_2, \cdots) \in l^1$ and fix $\epsilon > 0$. $\exists N \in \mathbb{N}$ such that $\sum_{n>N} 1/n^2 + \sum_{n>N} |x_n| < \epsilon$. Let a be large enough such that $a/N^3 \geqslant \max\{|x_1 - 1|, |x_2 - 1/2^2|, \cdots, |x_N - 1/N^2|\}$.

Then we have $y = (a, x_1, \cdots, x_N, 1/(N+1)^2, \cdots) - (a - x_0, 0, 0, \cdots) \in A - D$ and $||y - x|| < \epsilon$.

$$(a, x_1, \cdots, x_N, 1/(N+1)^2, \cdots) \in A$$

because

$$x_0 = a \geqslant \max_{n \leqslant N}\{N^3 |x_n - \beta_n/\alpha_n|\} \geqslant \max_{n \leqslant N}\{\alpha_n |x_n - \beta_n/\alpha_n|\}$$

and

$$|x_n - \beta_n/\alpha_n| = 0, \text{if} \quad n > N$$

$||y - x|| < \epsilon$ because

$$y - x = (0, 0, \cdots, 0, 1/(N+1)^2 - x_{N+1}, 1/(N+2)^2 - x_{N+1}, \cdots)$$

$$||y - x|| \leqslant \sum_{n>N} 1/n^2 + \sum_{n>N} |x_n| < \epsilon$$

So A, D can not be separated. Otherwise there exists a hyperplane separates $A - D$ and 0. Since $A - D$ is dense, it's not possible. □

Remark 8. *1. A is a convex, closed set with empty interior point. Let $B(x, \epsilon)$ be a small ball in l^1, where $x = (x_0, x_1, x_2, \cdots)$. Let N be large enough such that $1/N^2 + x_0/N^3 < \epsilon/4$ and $|x_N| < \epsilon/4$, then $(x_0, x_1, \cdots, x_N + \epsilon/2, x_{N+1}, \cdots) \in B(x, \epsilon) \setminus A$.*

$$(x_0, x_1, \cdots, x_N + \epsilon/2, x_{N+1}, \cdots) \notin A$$

Because: $|x_N| < \epsilon/4$, so

$$|x_N + \epsilon/2| > \epsilon/2 - \epsilon/4 > \epsilon/4 > 1/N^2 + x_0/N^3$$

$$x_0 < N^3(|x_N + \epsilon/2| - 1/N^2) \leqslant N^3 |x_N + \epsilon/2 - 1/N^2| = |\alpha_N(x_N + \epsilon/2) - \beta_N|$$

Note:The reason A has no interior point is that there is no uniform range for x_n.

2. In this example, any two positive sequence $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ satisfying

(i) $\beta_n \longrightarrow \infty$

(ii) $\{\beta_n/\alpha_n\}$ summable

would be enough.

3. *Moreover, if we replace* l^1 *by* l^p *space and let* $\beta_n \longrightarrow \infty$ *and* $\{|\beta_n/\alpha_n|^p\}$ *be summable, then we have two closed convex sets which can not be separated in* l^p *space.*

Next is another example of closed convex set with empty interior in Hilbert space.

Example 9. *Let* H *be a separable Hilbert space over* $\mathbb{R}$ *with an orthogonal basis* $\{e_n\}_{n=1}^{\infty}$*. Let* A *be the smallest closed convex balanced set containing all* e_n/n*. There exist lines intersecting* A *at only* 0*. There is no interior point of* A*. (See [2] page V.71 ex 10)*

Definition 10. X *is a topological vector space over the scalar field* $\Phi = \mathbb{R}$ *or* $\mathbb{C}$ *. A set* $E \subset X$ *is balanced if* $\forall\ \alpha \in \Phi$ *and* $|\alpha| \leqslant 1$, $\alpha E \subset E$*.*

References

[1] Marta Lewicka. Lecture notes on Real Analysis. Spring 2007, University of Minnesota;

[2] N. Bourbaki, *Topological Vector Space.*

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