

Homework 1

1. Let E be a linear space.
 - (i) Let $\|\cdot\| : E \rightarrow \mathbf{R}$ be a norm in E . Define $d : E \times E \rightarrow \mathbf{R}$ by: $d(x, y) = \|x - y\|$. Prove that (E, d) is a metric space.
 - (ii) Is every metric d in E induced by some norm $\|\cdot\|$, as in (i)?
 - (iii) The same question as in (ii) but assume that $E = \mathbf{R}$ and that d is NOT a multiple of the discrete metric.

Recall that the discrete metric $d : X \times X \rightarrow \mathbf{R}$ on a set X is given by: $d(x, y) = 0$ for $x = y$ and $d(x, y) = 1$ for $x \neq y$.

2. Prove that all norms in \mathbf{R}^n are equivalent.
3. Check if the following linear spaces E are normed by the given function $\|\cdot\| : E \rightarrow \mathbf{R}$. Are they Banach spaces?
 - (i) $E = \mathbf{R}^n$, $\|x\| = (\sum_{i=1}^n |x_i|^r)^{1/r}$, where $r \geq 1$ is a given number.
 - (ii) $E = l_2$, which are all sequences $x = \{x_i\}_{i=1}^\infty$ of real numbers that are square-summable: $\sum_{i=1}^\infty |x_i|^2 < \infty$. $\|x\| = (\sum_{i=1}^\infty |x_i|^2)^{1/2}$.
 - (iii) $E = l_2$ as in (ii), but: $\|f\| = \sup_{i \geq 1} |x_i|$.

4. The same question as in problem 3.
 - (i) $E = B(X)$, which are all bounded functions f from a set X to \mathbf{R} . $\|f\| = \sup\{|f(x)|; x \in X\}$.
 - (ii) $E = l_\infty$, which are all bounded sequences $x = \{x_n\}_{n=1}^\infty$ of real numbers. $\|x\| = \sup_{n \geq 1} |x_n|$.
 - (iii) $E = C([-1, 1]) \cap C^1((-1, 1))$, which are all continuous real functions f on the interval $[-1, 1]$, that are continuously differentiable in $(-1, 1)$. $\|f\| = \sup\{|f(x)|; x \in [-1, 1]\}$.

5. Let (X, d) be a metric space. Fix a reference point $x_0 \in X$ and let E be the vector space of all the Lipschitz continuous functions $f : X \rightarrow \mathbf{R}$ such that $f(x_0) = 0$. Define $\|f\|$ to be the smallest Lipschitz constant of f , that is:

$$\|f\| := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Prove that $(E, \|\cdot\|)$ is a Banach space.

6. (i) Let E be a normed space and E_0 its linear subspace. Let $T_0 : E_0 \rightarrow \mathbf{R}$ be a given linear functional. Show that there exists a linear functional $T : E \rightarrow \mathbf{R}$ such that $T|_{E_0} = T_0$.
 - (ii) Let E be an infinitely dimensional normed space (which means that there exists an infinite sequence of its elements, which are linearly independent). Use part (i) to prove that there exists a discontinuous linear functional $T : E \rightarrow \mathbf{R}$.