Marta Lewicka, Math 8601, Fall 2006

Homework 10

1. Prove that for every set $A \subset \mathbf{R}^n$ which is not of Lebesgue measure 0, there holds:

$$\forall c \in (0,1) \quad \exists P \subset \mathbf{R}^n \qquad \mu^*(A \cap P) > c\mu^*(P).$$

(P is a closed box and μ^* denotes the exterior Lebesgue measure).

2. Let $A \subset \mathbf{R}$ be a Lebesgue measurable set of positive measure. Prove that the following set:

$$A - A := \{x - y;\ x, y \in A\}$$

contains an interval $(-a, a)$, for some $a > 0$.
(This is called the Steinhaus Theorem. Hint: use the result of the previous exercise.)

3. Find a function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ whose set of points of discontinuity is:

(i) a given closed subset of $\mathbf{R}^n$,
(ii) a given F_σ subset of $\mathbf{R}^n$.

4. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a given measurable function (with respect to the Lebesgue measure). Are the following statements true?:

(i) $f^{-1}(A) \in \mathcal{L}_1$, for all $A \in \mathcal{L}_1$,
(ii) $f(A) \in \mathcal{L}_1$, for all $A \in \mathcal{L}_1$.

5. Prove that $\mathcal{L}_{n+m}$ is the smallest σ-algebra of subsets of $\mathbf{R}^{n+m}$, containing the product σ-algebra $\mathcal{L}_n \otimes \mathcal{L}_m$ and all sets of zero outer (Lebesgue) measure.

6. Is the following statement about the Borel σ-algebras true?:

$$\mathcal{B}(\mathbf{R}^{n+m}) = \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m).$$

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $A \in \mathcal{M}$.

(i) Assume that the sequence of integrals $\int_A |f_n|\ \mathrm{d}\mu$ is bounded and that the series of numbers $\sum_n |a_n|$ converges. Prove that the functional series $\sum_n |a_n f_n|$ converges μ-almost everywhere.
(ii) Prove that if $\int_A f_n^2\ \mathrm{d}\mu$ is a bounded sequence then the sequence of functions f_n/n converges to 0 μ-almost everywhere.