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Homework 11

1. Construct a strictly increasing function $f : [a, b] \longrightarrow \mathbf{R}$ whose set of points of discontinuity is a given countable subset of $[a, b]$.

2. Is the thesis of the Steinhaus theorem (problem 2 homework 10) still true if we replace $A - A$ by $A + A$? Does $A + A$ always contain some nondegenerate interval?

3. Prove that if $A \in \mathcal{L}_{n+m}$ has measure 0, then for almost every $x \in \mathbf{R}^n$ the set:

$$A_x := \{y \in \mathbf{R}^k;\ (x, y) \in A\}$$

is in $\mathcal{L}_m$ and has measure 0.

4. Using the result of problem 3 and the Fubini-Tonelli theorem for product measures deduce the following Fubini-Tonelli theorem for the Lebesgue integral.

Let $f : \mathbf{R}^{n+m} \longrightarrow \overline{\mathbf{R}}$ be a nonnegative (Lebesgue) measurable or a (Lebesgue) integrable function. Then:

(i) For almost every $x \in \mathbf{R}^n$ the function $y \mapsto f(x, y)$ is measurable and its integral $\int_{\mathbf{R}^m} f(x, \cdot)\ \mathrm{d}\mu_m$ is well defined.

(ii) The function

$$x \mapsto \begin{cases} \int_{\mathbf{R}^m} f(x, \cdot)\ \mathrm{d}\mu_m & \text{when defined} \\ 0 & \text{elsewhere} \end{cases}$$

is measurable and its integral over $\mathbf{R}^n$ is well defined.

(iii) One has the following formula:

$$\int_{\mathbf{R}^{n+m}} f\ \mathrm{d}\mu_{m+n} = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} f\ \mathrm{d}\mu_m \right)\ \mathrm{d}\mu_n.$$

5. Let U be a bounded open subset of $\mathbf{R}^n$ and let $f : (a, b) \times U \longrightarrow \mathbf{R}$ be a continuous function such that for each $(t, x) \in (a, b) \times U$ the partial derivative $\partial f/\partial t\ (t, x)$ exists and satisfies: $\|\partial f/\partial t\ (t, x)\| \leq g(x)$, for some integrable function $g : U \longrightarrow \mathbf{R}$. Define the function: $F(t) := \int_U f(t, \cdot)\ \mathrm{d}\mu_n$. Prove that F is differentiable and that:

$$F'(t) = \int_U \frac{\partial f}{\partial t}(t, \cdot)\ \mathrm{d}\mu_n.$$

6. For given vectors $v_1, \ldots, v_n$ in $\mathbf{R}^n$ define the block:

$$B = \left\{ \sum_{i=1^n} t_i v_i; \quad t_i \in [0, 1] \right\}.$$

Prove that the Lebesgue measure of B equals to the absolute value of the determinant of the matrix whose columns are vectors $v_1, \ldots, v_n$.

Hint: Consider the function $f(v_1, \ldots, v_n) = \mu(B) \cdot sign(\det[v_1, \ldots, v_n])$. Prove first that f is n-linear, anti-symmetric (so that permuting any two of the vectors v_i changes the sign of f), and equals 1 on the Euclidean basis. Then prove that such a function must necessarily be the determinant of the matrix $[v_1, \ldots, v_n]$.

7. Find the volume of the intersection of the following two subsets of $\mathbf{R}^3$:

$$C_1 = \{(x, y, z);\ x^2 + y^2 \leq 1\}, \qquad C_2 = \{(x, y, z);\ y^2 + z^2 \leq 1\}.$$