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Homework 12

1. Given an open box $A = (a_1, b_1) \times \ldots \times (a_n, b_n) \subset \mathbf{R}^n$ of finite, positive volume, construct a sequence ϕ_n of functions in $\mathcal{C}_c^\infty(\mathbf{R}^n, \mathbf{R})$ such that: $\phi_n \nearrow \mathbf{1}_A$ (the convergence is pointwise everywhere).

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, so that $\mu(X) < \infty$. We say that a sequence of real-valued, integrable functions f_n on X is *uniformly integrable*, if $\sup_n \{\int_X |f_n| \, \mathrm{d}\mu\} < \infty$ and:

$$\forall \epsilon > 0 \quad \exists \delta > 0 \qquad \mu(A) < \delta \implies \forall n \quad \int_A |f_n| \, \mathrm{d}\mu < \epsilon.$$

Prove that a sequence f_n satisfies $\int_X |f_n - f| \, \mathrm{d}\mu \to 0$ if and only if both f_n converges to f in measure and the f_n are uniformly integrable.

3. Let $f : [0,1] \longrightarrow \mathbf{R}$ be a continuous function. For each $c \in \mathbf{R}$ let $n(c)$ denote the number of solutions of the equation $f(x) = c$. Prove that the function $n : \mathbf{R} \longrightarrow \bar{\mathbf{R}}_+$ is Lebesgue measurable.

4. Let $f : [0, \pi] \longrightarrow \mathbf{R}$ be given by:

$$f_n(x) = n \frac{\sin x}{1 + n^2 \sin^2 x}.$$

For a given $\epsilon > 0$, find explicitly the Egoroff set E_ϵ on which the sequence f_n converges uniformly, and such that $\mu(E_\epsilon) > \pi - \epsilon$.

5. Prove that every (Lebesgue) measurable function $f : [0,1] \longrightarrow \mathbf{R}$ is a limit almost everywhere of a sequence $\{f_n\}$ of continuous functions. Is it always possible to choose this sequence to be monotone?

6. Let $f, g : (0,1) \longrightarrow [0, \infty)$ be (Lebesgue) measurable functions. Prove that if

$$\forall \alpha > 0 \qquad \mu\{x \in (0,1), \ g(x) > \alpha\} \leq \int_{\{x \in (0,1); \ f(x) > \alpha\}} f(x) \, \mathrm{d}x,$$

then:

$$\int_0^1 g(x)^p \, \mathrm{d}x \leq \int_0^1 f(x)^{p+1} \, \mathrm{d}x$$

for every $p > 0$.

Prove also that if "$\leq$" is replaced in both places above by "$\geq$" then the result is still true.

7. Prove that for any (Lebesgue) measurable function $f : (0,1) \longrightarrow (0, \infty)$ there holds:

$$\int_{[0,1]} f \, \mathrm{d}\mu \cdot \int_{[0,1]} 1/f \, \mathrm{d}\mu \geq 1.$$