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Homework 13

1. Let $\mathcal{F}$ be a family of real valued continuous functions on a complete metric space X. Assume that:

$$\forall x \in X \quad \exists M \quad \forall f \in \mathcal{F} \qquad |f(x)| \leq M.$$

Prove that there exists a nonempty open set $U \subset X$ and a constant N such that:

$$\forall f \in \mathcal{F} \quad \forall x \in U \qquad |f(x)| \leq N.$$

2. Show that the following subset of the Banach space c_0:

$$\Big\{ \{x_n\}_{n=1}^{\infty};\ \sup_n n|x_n| < \infty \Big\}$$

is of first Baire's category.

3. Give an example of a discontinuous linear map between normed spaces, so that:

(i) its graph is closed and its target space is Banach,

(ii) its graph is closed and its domain space is Banach.

4. Let E be a Banach space.

(i) Prove that if x_n converges weakly to x in E and T_n converges (strongly) to T in E^* then $T_n(x_n)$ converges to $T(x)$.

(ii) Prove that if x_n converges (strongly) to x in E and T_n converges weakly * to T in E^* then $T_n(x_n)$ converges to $T(x)$.

(iii) What if x_n converges weakly to x and T_n converges weakly * to T?

5. Let E be a normed space. Let $\mathcal{A}_0$ be the collection of sets U of the form:

$$U = \{x \in E;\ |T_i(x)| < \epsilon_i,\ i : 1 \ldots n\} = \bigcap_{i=1}^{n} T_i^{-1}(-\epsilon_i, \epsilon_i)$$

for some $n \geq 1$ and $\epsilon_i > 0$, $T_i \in E^*$, $i : 1 \ldots n$.

For a given $x_0 \in E$, define the family of translated sets U:

$$\mathcal{A}_{x_0} = \{x_0 + U;\ U \in \mathcal{A}_0\}.$$

(i) Prove that the weak topology on E is composed of the empty set and (arbitrary) unions of sets from the family: $\bigcup_{x_0 \in E} \mathcal{A}_{x_0}$.

(ii) Prove that $\mathcal{A}_{x_0}$ is a family of open neighbourhoods of x_0.

6. Let E be a normed space. Prove that the weak * topology on E^* is given through the same procedure as in problem 5 (translations and arbitrary unions), applied to the basis of open neighbourhoods of 0, $\mathcal{A}_0$, containing sets U of the form:

$$U = \{T \in E^*;\ |T(x_i)| < \epsilon_i,\ i : 1 \ldots n\} = \bigcap_{i=1}^{n} J(x_i)^{-1}(-\epsilon_i, \epsilon_i)$$

for some $n \geq 1$ and $\epsilon_i > 0$, $x_i \in E$, $i : 1 \ldots n$.

Here $J(x_i) \in E^{**}$ is the operator of evaluation on x_i.