Marta Lewicka, Math 8602, Spring 2007

Homework 14

1. Prove that if E is a reflexive Banach space then in E^* the weak topology coincides with the weak * topology.

2. Let E be a finitely dimensional normed space. Prove that:

(i) The strong and the weak topology in E coincide.
(ii) A sequence in E converges weakly iff it converges strongly.

3. Let E be a normed space. Prove that a sequence $T_n \in E^*$ converges to $T \in E^*$ in the weak * topology if and only if T_n converges to T weakly *.

4. Let E, F be two Banach spaces. Let T be a linear operator from E to F. Prove that the following statements are equivalent:

(i) $T \in \mathcal{L}(E, F)$.
(ii) T is continuous from E with weak topology to F with weak topology.

5. Based on the first separation theorem from class, prove the following geometric form of the Hahn-Banach theorem.

Let E be a normed space and let A, B be two nonempty, convex, disjoint subsets of E. Assume that one of the following conditions hold:

(i) at least one of the sets A, B is open,
(ii) both sets A, B are closed and at least one of them is compact.

Then there exists $T \in E^*$ and $\alpha \in \mathbf{R}$ such that:

$$T(x) \leq \alpha \leq T(y) \qquad \forall x \in A \quad \forall y \in B.$$

6. Using the theorem from problem 5, prove the following distance formula.

Let E_0 be a linear subspace of a normed space E. For $x_0 \in E$, define the distance:

$$d(x_0, E_0) = \inf\{||x_0 - x||;\ x \in E_0\}.$$

Define the set of functionals anihilating E_0:

$$E_0^\perp := \{T \in E^*;\ T_{|E_0} = 0\}.$$

Then:

$$d(x_0, E_0) = \sup\{|T(x_0)|;\ T \in E_0^\perp \text{ and } ||T|| \leq 1\}.$$

Notice that when $E_0 = \{0\}$ then we have known this result before.