Marta Lewicka, Math 8602, Spring 2007

Homework 15

1. Prove the following geometrical form of the Hahn-Banach theorem.

Let E be a normed space and let W be its open, convex, nonempty subset. Let F be a linear subspace of E, such that $W \cap F = \emptyset$. Then there exists a hyperspace H of E (that is, $H = T^{-1}(0)$ for some linear, nonzero functional $T : E \longrightarrow \mathbf{R}$) which contains F and such that $W \cap H = \emptyset$. Moreover, H with these properties is automaticaly closed.

[Hint: First observe that the theorem is valid for $E = \mathbf{R^2}$. In the general setting, use Zorn's lemma to construct H.]

2. Let E be a Banach space. Prove that the weak* topology on E^* is metrizable iff E is finitely dimensional.

3. Let E be a normed space. Prove that if $\phi \in E^{**} \setminus J(E)$ then ϕ is not continuous with respect to the weak* topology on E^*. Deduce that the weak and the weak* topology on E^* coincide iff E is reflexive.

[Hint: Prove first that if $\phi : E^* \longrightarrow \mathbf{R}$ is linear and continuous with respect to the weak* topology (on E^*), then there exist $x_1, x_2 \dots, x_n \in E$ such that:

$$\bigcap_{i=1}^{n} Ker(J(x_i)) \subset Ker(\phi).$$

Then prove that necessarily ϕ must be a linear combination of $J(x_i)$.]

4. Show that the unit closed ball in l_∞^* is weak* compact but it is not sequentially compact with respect to the weak* topology.

5. Prove Schur's Lemma: the weak and strong convergence in l_1 coincide.

The proof may be divided in the following steps:

(i) If the lemma was not correct, we would have a sequence $\{\{x_n^k\}_{n=1}^\infty\}_{k=1}^\infty$ converging weakly to 0 in l_1 (as $k \to 0$) and such that:

$$\|\{x_n^k\}_{n=1}^\infty\|_{l_1} \geq 1 \qquad \forall k.$$

(ii) One could find then an increasing sequence of natural numbers $\{n_k\}_{k=1}^\infty$ such that (without loss of generality):

$$\sum_{n=n_k}^{n_{k+1}-1} |x_n^k| > \frac{3}{4} \|\{x_n^k\}_{n=1}^\infty\|_{l_1} \qquad \forall k.$$

(iii) The above contradicts the weak convergence of our sequence to 0.

6. Prove that any subspace of a separable Banach space is also separable.