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Homework 16

1. Prove that l_2 is a Hilbert space, and that the following spaces are not Hilbert: l_1, l_∞, $\mathcal{C}(K)$ (space of continuous functions on some compact subset metric space, with the sup norm).

2. Deduce the Riesz-Frechet theorem from the projection theorem.

[Hint: Given a nonzero $T \in H^*$ think of the closed subspace $M = T^{-1}(0)$. There exists $x \in M^\perp$ with unit norm (why?). Then there should be $T(\cdot) = \langle T(x)x, \cdot\rangle$.]

3. Without using the theory of uniformly convex spaces, prove elementary, that in a Hilbert space the strong convergence is equivalent to the weak convergence and convergence of the norms.

4. Let H be a Hilbert space and let E be its linear subspace. Recall the definition of functionals annihilating a subspace (see problem 6 homework 14) and define:

$$E^{\perp\perp} := J^{-1}\big((E^\perp)^\perp\big).$$

(i) Prove that $E^{\perp\perp}$ is the closure of E in H.

(ii) Let A be an arbitrary subset of H. We may define:

$$A^\perp = \{x \in H;\ \langle x, a\rangle = 0 \quad \forall a \in A\}.$$

What is $(A^\perp)^\perp$?

5. Let E be a Banach space and let $F \neq E$ be its closed subspace.

(i) Prove that for every $\epsilon > 0$ there exists $x_\epsilon \in E$ of norm 1 and such that $\mathrm{dist}(x_\epsilon, F) \geq (1 - \epsilon)$. (We say that x_ϵ is ϵ-perpendicular to F).

(ii) Using (i), prove that the closed unit ball in E is compact (in the strong topology) iff E has finite dimension.

6. Prove that every uniformly convex space is strictly convex (recall the definition from last semester and where we used it)