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Homework 17

1. Prove the following generalization of the Holder inequality. Let $1 \leq p_i \leq \infty$, where $i : 1..k$ satisfy $\sum_{i=1}^k 1/p_i = 1/p$ for some $p \in [1, \infty]$. If $f_i \in L^{p_i}(\Omega)$, $i : 1 \dots k$, then the product $f_1 ... f_n$ belongs to the space $L^p(\Omega)$ and:

$$||f_1 ... f_n||_{L^p} \leq ||f_1||_{L^{p_1}} ... ||f_n||_{L^{p_n}}.$$

2. Prove the following interpolation inequality. If $f \in L^p(\Omega) \cap L^q(\Omega)$, for some $1 \leq p \leq q \leq \infty$, then $f \in L^r(\Omega)$ for any $r \in [p, q]$ and there holds:

$$||f||_{L^r} \leq ||f||_{L^p}^{\alpha} ||f||_{L^q}^{1-\alpha}, \qquad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \ \alpha \in [0, 1].$$

3. Suppose that $\Omega \subset \mathbf{R}^n$ has finite measure and let $1 \leq p, q \leq \infty$.

(i) Prove that if $f \in L^q(\Omega)$ and $p \leq q$, then $f \in L^p(\Omega)$ and:

$$||f||_{L^p} \leq (\mu(\Omega))^{1/p - 1/q} ||f||_{L^q}.$$

(ii) Prove that for any pair of distinct exponents $p \neq q$ we have: $L^p(\mathbf{R}) \not\subset L^q(\mathbf{R})$. Hint: consider the function:

$$f(x) = \frac{1}{|x|^{1/2} \sqrt{1 + \log^2 |x|}}$$

4. Prove or disprove the following statement. Every Banach space in which the parallelogram identity holds is a Hilbert space (in the sence that it admits a scalar product which induces its norm).

5. Use the following outline to prove that the unit ball $\overline{B}_{E^*}$ in E^* (E is a Banach space) with weak * topology is metrizable iff E is separable.

Proof of separability $\Longrightarrow$ metrizability:

(i) Find a sequence x_n of elements in $\overline{B}_E$, dense in this ball, and define:

$$d(T, S) := \sum_{n=1}^{\infty} \frac{1}{2^n} |(T - S)(x_n)| \qquad \forall T, S \in \overline{B}_{E^*}.$$

Check that d is a metric on $\overline{B}_{E^*}$.

(ii) Take a basic weak * open neighbourhood U of T in $\overline{B}_{E^*}$, given through evaluations at points $y_1, \dots y_k$ in $\overline{B}_E$. Approximate each y_i by a x_{n_i} and choose r much smaller than each 2^{-n_i}. The open ball centered at T and of radius r with respect to the metric d should then be contained in U.

(iii) Conversly, think of an open ball centered at T and of radius $r > 0$ with respect to the metric d. Construct its subset which is basic weak * open neighbourhood of T in $\overline{B}_{E^*}$. The convergence of the series in the definition of d is a hint.

Proof of mertizability $\Longrightarrow$ separability:

(iv) Consider a decreasing sequence of open balls B_n centered at 0 and of radii say $1/n$ with respect to the metric. Each B_n contains a basic weak * open neighbourhood of 0 in $\overline{B}_{E^*}$, given through evaluations at a finite collection of points $A_n \subset \overline{B}_E$. Take $D = \bigcup A_n$. The subspace $F = \mathrm{span}(D)$ is dense in E because $F^{\perp} = \{0\}$.

6. A function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ is called a Borel function if $f^{-1}((-\infty, a)) \in \mathcal{B}_n$ for every $a \in \mathbf{R}$. Is the following statement true?: Every measurable function f becomes a Borel function after suitable modification on a set of Lebesgue measure zero.