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Homework 18

1. For $p \in (0,1)$, define $L^p[0,1]$ and $||\cdot||_{L^p}$ by the standard formula. Show that $||\cdot||_{L^p}$ is not a norm.

2. Let $f \in L^p(\mathbf{R}^n)$, $g \in L^q(\mathbf{R}^n)$, $p,q \in [1,\infty]$ and $1/p + 1/q - 1 = 1/r \geq 0$. Prove that $f * g \in L^r(\mathbf{R}^n)$ and $\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$.

3. Let Ω be an open, bounded subset of $\mathbf{R}^n$. Prove that for every $f \in L^\infty(\Omega)$:

$$\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}.$$

4. Let $p,q \in [1,\infty]$. For which functions f the operator of multiplication by f is continuous from $L^p[0,1]$ to $L^q[0,1]$?

5. Let T be a linear function between two Banach spaces E_1 and E_2. Assume that T takes every strongly convergent sequence into a weakly convergent sequence. Prove that $T \in \mathcal{L}(E_1, E_2)$.

6. Prove that every Banach space E is linearly isometric to a closed subspace of $\mathcal{C}(X)$, where X is a compact topological space. When E is separable, prove that X may be taken as the interval $[0,1]$.

[Hint: If X is a compact metric space which can be seen as a convex subset of some linear space, then show that there exists a continuous function f from $[0,1]$ onto X.]