Marta Lewicka, Math 8602, Spring 2007

Homework 19

1. Let Ω be a bounded subset of $\mathbf{R}^n$ and let S be a linear subspace of $L^\infty(\Omega)$. Assume that:

$$\exists C > 0 \quad \forall f \in S \qquad \|f\|_{L^\infty} \leq C\|f\|_{L^2}.$$

Prove that S must be finitely dimensional and that its dimension is at most C^2.

2. Define the space $\mathcal{C}_0(\mathbf{R}^n)$, composed of all functions $f \in \mathcal{C}(\mathbf{R}^n)$ vanishing at infinity, that is such that:

$$\forall \epsilon > 0 \quad \exists n \quad \forall \|x\| \geq n \qquad |f(x)| < \epsilon.$$

Prove that $\mathcal{C}_0(\mathbf{R}^n)$ is the completion of $\mathcal{C}_c(\mathbf{R}^n)$ in the L^∞ norm.

3. Let $1 < p < \infty$. For every $f \in L^p((0,\infty))$, define:

$$F(x) = \frac{1}{x}\int_0^x f(t)\ \mathrm{d}t \qquad \forall x \in (0,\infty).$$

Prove that $F \in L^p((0,\infty))$ and:

$$\|F\|_{L^p} \leq \frac{p}{p-1}\|f\|_{L^p}.$$

[Hint: Assume first that f is nonnegative and compactly supported in $(0,\infty)$. Integrate by parts and notice that $xF'(x) = f(x) - F(x)$.]

4. Using Jensen's inequality, prove:

(i) Holder's inequality [Hint: Use $\phi(x) = x^p$ and consider first the case of f, g nonnegative and such that $\|g\|_{L^q} = 1$.]

(ii) Minkowski's inequality (that is, the triangle inequality for the norm in L^p [Hint: Use $\phi(x) = (1 - x^{1/p})^p$ and assume first that f, g are nonnegative, $f \leq 1$ and $\|f + g\|_{L^p} = 1$.]

5. Let $f \in L^\infty(\Omega)$, where Ω is an open subset of $\mathbf{R}^n$. Let ρ_n be a sequence of mollifiers in $\mathbf{R}^n$. Is it true that $\rho_n * f$ converges to f in $L^\infty(\Omega)$?

6. For which functions $f \in L^p$ and $g \in L^{p'}$ we have 'equality' in Holder's inequlaity?:

$$\|fg\|_{L^1} = \|f\|_{L^p}\|g\|_{L^{p'}}$$