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Homework 2

1. Let s be the linear space of all sequences of real numbers.

(i) Prove that the function $d : s \times s \longrightarrow \mathbf{R}$:

$$d(\{x_i\}, \{y_i\}) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

is a metric on s, and that the metric space (s, d) is complete.

(ii) Prove that every open neighbourhood of 0 in s contains the whole line $\{\alpha x;\ \alpha \in \mathbf{R}\}$, for some $x \in s \setminus \{0\}$.

(iii) Deduce from (ii) that there is no norm in s, which would make it a Banach space, topologically equivalent to (s, d) (which means, with the convergence of sequences in this norm the same as the convergence in the metric space (s, d)).

2. Let E be a linear space and let $\|\cdot\|$ and $\|\cdot\|_1$ be two norms on E.

(i) Assume that $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Prove that $(E, \|\cdot\|)$ is a Banach space iff $(E, \|\cdot\|_1)$ is a Banach space.

(ii) Give an example of E, $\|\cdot\|$ and $\|\cdot\|_1$ so that $(E, \|\cdot\|)$ is a Banach space but $(E, \|\cdot\|_1)$ is not.

(iii) Prove that the function $\|\cdot\| : E \longrightarrow \mathbf{R}$ is continuous on $(E, \|\cdot\|)$. Must it be continuous on $(E, \|\cdot\|_1)$ as well?

3. Find the norms of the following linear functionals on $\mathcal{C}[a, b]$ (with the norm of the uniform convergence $\|f\| = \max\{|f(x)|;\ x \in [a, b]\}$):

(i) $T(f) := \int_a^b f(x)\ \mathrm{d}x$,

(ii) $T_g(f) := \int_a^b f(x)g(x)\ \mathrm{d}x$, where g is a fixed element of $\mathcal{C}[a, b]$,

(iii) $T(f) := \sum_{i=1}^n \lambda_i \cdot f(x_i)$, where $x_1 ... x_n \in [a, b]$ and $\lambda_1 ... \lambda_n \in \mathbf{R}$ are given parameters.

4. Let E be a normed space and let $T \in E^*$. Does there necessarily exist $x \in E$ such that $\|x\| = 1$ and $|T(x)| = \|T\|$?

5. We say that a normed space E is strictly convex iff:

$$\forall x, y \in E \quad \forall t \in (0, 1) \quad \|x\| = \|y\| = 1 \implies \|tx + (1 - t)y\| < 1.$$

Prove that if E^* is strictly convex then for every $x_0 \in E$ there exists EXACTLY ONE functional $T \in E^*$ such that $\|T\| = \|x_0\|$ and $T(x_0) = \|x_0\|^2$.

6. Let E, F be two normed spaces. Prove that if $\mathcal{L}(E, F)$ is a Banach space, then F is also Banach. [Hint: First assume that $E = \mathbf{R}$.]