Marta Lewicka, Math 8602, Spring 2007

Homework 20

1. Let $y_1, \dots, y_n$ and $\alpha_1, \dots, \alpha_n$ be positive real numbers such that $\sum_{i=1}^n \alpha_i = 1$. Prove that:

$$\prod_{i=1}^n y_i^{\alpha_i} \leq \sum_{i=1}^n \alpha_i y_i.$$

2. Prove the converse of the Riesz - Frechet - Kolmogorov theorem. Assume that a subset $\mathcal{F}$ of $L^p(\mathbf{R}^n)$ has compact closure. (We assume that $1 < p < \infty$.) Then there holds:

(i) $\exists C > 0 \quad \forall f \in \mathcal{F} \qquad \|f\|_{L^p(\mathbf{R}^n)} \leq C,$

(ii) $\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall h \in B(0, \delta) \quad \forall f \in \mathcal{F} \qquad \|\tau_h f - f\|_{L^p(\mathbf{R}^n)} < \epsilon,$

(iii) $\forall \epsilon > 0 \quad \exists \Omega \in \mathcal{L}_n$, bounded $\quad \forall f \in \mathcal{F} \qquad \|f\|_{L^p(\mathbf{R}^n \setminus \Omega)} < \epsilon.$

3. Let $\{a_n\}$ is a sequence of positive numbers. Prove the following Hardy's inequality:

$$\forall 1 < p < \infty \qquad \sum_{N=1}^\infty \left(\frac{1}{N} \sum_{n=1}^N a_n \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^\infty a_n^p.$$

[Hint: If the sequence $\{a_n\}$ is nonincreasing, then use the result in problem 3 Homework 19.]

4. Prove that every convex function $f : (a, b) \longrightarrow \mathbf{R}$ must be continuous.

5. Let $g \in L^1(\mathbf{R}^n)$ and let A be a bounded subset of some $L^p(\mathbf{R}^n)$, $1 < p < \infty$. Define $\mathcal{F} = \{g * f;\ f \in A\}$. Prove that $\mathcal{F}$ has compact closure in $L^p(\Omega)$, for any $\Omega \subset \mathbf{R}^n$ of finite measure.

6. Let $\Omega \subset \mathbf{R}^n$ be open, bounded and of (Lebesgue) measure 1. Let $f \in L^1(\Omega)$. Prove that:

$$\lim_{p \to 0} \left(\int_\Omega |f|^p \right)^{1/p} = \exp \left(\int_\Omega \ln |f| \right).$$

[Hint: Use Jensen's inequality to prove $\leq'$. For the converse inequality, notice that $\ln |x| \leq |x| - 1$ and $\lim_{p \to 0} (|x|^p - 1)/p = \ln |x|$.]

7. The definition of Lebesgue points applies to individual integrable functions and not to equivalence classes. Otherwise, call a point $x \in \mathbf{R}^n$ a Lebesgue point of the equivalence class of $f \in L^1(\mathbf{R}^n)$ if there exists $a \in \mathbf{R}$ such that

$$\lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B(x,r)} |f - a| = 0.$$

(i) Prove that if the above formula holds (for some a), then it holds for every function $\tilde{f}$ in the equivalence class of f (and with the same a).

(ii) Define $F(x)$ to be equal to a when x is a Lebesgue point (of the equivalence class of f), and 0 otherwise. Prove that F is in the equivalence class of f. That is, if x is a Lebesgue point of f (as an individual function) then x is also a Lebesgue point of the equivalence class of f and $f(x) = F(x)$.

The function F is thus a member of the equivalence class of f that has a maximal set of Lebesgue points.