Marta Lewicka, Math 8602, Spring 2007

Homework 21

1. Prove that a measurable function $u : \Omega \longrightarrow \mathbf{R}$ belongs to $L^p(\Omega)$ iff:

$$\sup \left\{ \int_\Omega |u(x)|v(x)\ \mathrm{d}x;\ v(x) \geq 0 \text{ on } \Omega, \text{ and } \|v\|_{L^{p'}(\Omega)\leq 1} \right\} < \infty$$

and then that supremum equals $\|u\|_{L^p(\Omega)}$.

2. Prove the following version of the Young's Theorem. Let $p, q, r \geq 1$ be such that $1/p + 1/q + 1/r = 2$. Then:

$$\left| \int_{\mathbf{R}^n} (u * v)w \right| \leq \|u\|_{L^p} \|v\|_{L^q} \|w\|_{L^r}$$

for every $u \in L^p(\mathbf{R}^n)$,$v \in L^q(\mathbf{R}^n)$, $w \in L^r(\mathbf{R}^n)$.

3. Let $p \in [1, \infty]$ and $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Prove that $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and that $\nabla(uv) = v\nabla u + u\nabla v$.

4. Let $p \in [1, \infty]$, $u \in W^{1,p}(\Omega)$ and let $f \in \mathcal{C}^\infty(\mathbf{R})$ be such that $f(0) = 0$ and $f' \in L^\infty$. Prove that the composition $f \circ u \in W^{1,p}(\Omega)$ and $\nabla(f \circ u) = (f' \circ u)\nabla u$.

5. Let $p \in [1, \infty]$, $u \in W^{1,p}(\Omega)$ and let $h : \tilde{\Omega} \longrightarrow \Omega$ be a diffeomorphism of class $\mathcal{C}^\infty$ between the open sets $\tilde{\Omega}$ and Ω. Assume that both det Dh and det $D(h^{-1})$ are in $\in L^\infty$. Prove that $u \circ h \in W^{1,p}(\tilde{\Omega})$ and $\nabla(u \circ h) = [(\nabla u) \circ h]Dh$.

6. Prove the following covering theorem. Let $\mathcal{F}$ be a family of open balls in some metric space (X, d), whose diameters are uniformly bounded. Then it must have a pairwise disjoint subfamily $\mathcal{G}$ such that:

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B.$$

where $5B$ denotes the ball with the same center as B and of radius 5 times bigger.

[Hint: Use Zorn's Lemma to the family of all pairwise disjoint subfamilies ω of $\mathcal{F}$, with the property: if a ball $B \in \mathcal{F}$ intersects some ball from ω then it must intersect another ball whose radius is $\geq$ half the radius of B.]