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Homework 22

1. Let $p \in [1, \infty]$. For $u \in W^{1,p}(\Omega)$, define $u^+ = \max(u, 0)$.

(i) Prove that $u^+ \in W^{1,p}(\Omega)$ and that

$$\nabla(u^+) = \begin{cases} \nabla u & \text{a.e. in } \{x;\ u(x) > 0\} \\ 0 & \text{a.e. in } \{x;\ u(x) \leq 0\}. \end{cases}$$

[Hint: You may think of using the result in problem 4 homework 21.]

(ii) Given $c \in \mathbf{R}$, show that $\nabla u = 0$ almost everywhere in the (measurable) set $\{x;\ u(x) = c\}$.

(iii) Prove that if the sequence $\{u_n\}$ converges to u in $W^{1,p}(\Omega)$ then also $\{u_n^+\}$ converges to u^+.

2. Let $p \in (1, \infty)$ and u_n be a bounded sequence in some $L^p(\Omega)$.

(i) Assume that u_n converges weakly to u. Must it converge pointwise a.e.?

(ii) Prove that if u_n converges pointwise a.e. to seome function u, then u_n converges to u weakly in $L^p(\Omega)$.

3. Consider the function $u(x) = |\log \|x\||^\alpha$. Prove that when $\alpha \in (0, 1-1/n)$ then:

$$u \in (W^{1,n} \setminus L^\infty)(B_{1/2} \subset \mathbf{R}^n).$$

4. Recall that for $k \geq 1$ we have an inductive definition::

$$W^{k,p}(\Omega) = \left\{u \in W^{k-1,p}(\Omega);\ \forall i : 1 \dots n \quad \partial u/\partial x_i \in W^{k-1,p}(\Omega)\right\}.$$

Using the established facts for $W^{1,p}(\Omega)$, prove that for every $k \geq 1$ one has:

(i) $W^{k,p}(\Omega)$ is a Banach space.

(ii) $W^{k,p}(\Omega)$ is reflexive, when $1 < p < +\infty$.

(iii) $W^{k,p}(\Omega)$ is separable, when $p \neq \infty$.

(iv) $W^{k,2}(\Omega)$ is a separable Hilbert space.

Let $\Omega \subset \mathbf{R}^n$ be open, bounded and of class $\mathcal{C}^1$. Let $u \in W^{k,p}(\Omega)$. Using the established facts for $W^{1,p}(\Omega)$, deduce the statements in problems 5 and 6:

5. If $k < n/p$ then $u \in L^q(\Omega)$, where $1/q = 1/p - k/n$ and for some constant C depending only on k, p, n and Ω one has:

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}.$$

6. If $k > n/p$ then $u \in \mathcal{C}^{k-[n/p]-1,\alpha}(\bar{\Omega})$, where $\alpha = [n/p] + 1 - n/p$ if n/p is not an integer and otherwise α is any number in $(0, 1)$. Moreover, one has:

$$\|u\|_{\mathcal{C}^{k-[n/p]-1,\alpha}(\bar{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)},$$

with some constant C depending only on k, p, n, α and Ω.

Here $[n/p]$ stands for the biggest integer which is strictly smaller than n/p. The space $\mathcal{C}^{m,\alpha}(\bar{\Omega})$ consists of functions which are Holder continuous with the exponent α, together with all their derivatives, up to the order m. The norm in this space is given by the sum of $\mathcal{C}^{0,\alpha}$ norms of all the derivatives (and the function itself).