Marta Lewicka, Math 8602, Spring 2007

Homework 23

1. Let $\Omega \subset \mathbf{R}^n$ be open, bounded and of class $\mathcal{C}^1$.

(i) Prove that $u \in W^{1,\infty}(\Omega)$ iff u is Lipschitz continuous. and that that the weak gradient of u coincides a.e. in Ω with the strong gradient of u.

(ii) Show that the equality of both spaces ($W^{1,\infty}(\Omega)$ and the space of Lipschitz continuous functions on Ω) may not hold, if we do not assume that Ω is of class $\mathcal{C}^1$.

2. Prove directly that if $u \in W^{1,p}(0,1)$, for some $p \in (1,\infty)$ then:

$$\forall \text{ a.e. } x, y \in (0,1) \quad |u(x) - u(y)| \leq |x-y|^{1-1/p} \left(\int_0^1 |u'|^p \right)^{1/p}.$$

3. Prove the following interpolation inequality:

$$\forall u \in H^2(\Omega) \cap H_0^1(\Omega) \quad \|\nabla u\|_{L^2} \leq C \|u\|_{L^2}^{1/2} \|\nabla^2 u\|_{L^2}^{1/2}$$

where the constant C depands only on the open set $\Omega \subset \mathbf{R}^n$.

4. Prove that $W^{n,1}(\mathbf{R}^n) \subset L^\infty(\mathbf{R})^n$ and that this embedding is continuous.

5. Let $\Omega \subset \mathbf{R}^n$ be open, bounded and $\mathcal{C}^1$ and let $p \in [1,\infty)$. Prove that there does not exist a continuous linear operator:

$$T : L^p(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that $Tu = u_{|\partial\Omega}$ for all $u \in \mathcal{C}^1(\bar{\Omega})$.

6. Let $\Omega \subset \mathbf{R}^n$ be an open and connected set. Prove that if $u \in W^{1,p}(\Omega)$ and $\nabla u = 0$ a.e. in Ω then, for some constant C one has:

$$u(x) = C \qquad \forall \text{ a.e. } x \in \Omega.$$