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Homework 24

1. Prove the following simple statements.

(i) Let $f \in L^1_{loc}(\mathbf{R}^n)$ be a nonnegative function. Then the following is a Radon measure, absolutely continuous with respect to the Lebesgue measure dx in $\mathbf{R}^n$:

$$\forall A \in \mathcal{B}_n \qquad \mu(A) := \int_A f \ \mathrm{d}x.$$

Find an example of a Radon measure on $\mathbf{R}^n$ which is not defined in the above way.

(ii) Let μ be a Radon measure on $\mathbf{R}^n$ and let $A \in \mathcal{B}_n$. Define:

$$\forall B \in \mathcal{B}_n \qquad (\mu\lfloor A)(B) := \mu(A \cap B).$$

Then $\mu\lfloor A$ is a Radon measure, absolutely continuous with respect to μ.

2. Prove that any (nonnegative) Borel mesure on $\mathbf{R}^n$ which is locally finite and outer regular must be a Radon measure.

3. Let μ, ν be two Radon mesures on $\mathbf{R}^n$. Consider the following two conditions:

(i) $\nu << \mu$,

(ii) $\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall A \in \mathcal{B}_n \qquad \mu(A) < \delta \implies \nu(A) < \epsilon.$

Are these conditions equivalent? If yes, provide a proof, otherwise a counterexample. What if we additionally assume that ν is finite?

4. Prove uniqueness of the Lebesgue decomposition of (nonnegative) Radon measures.

5. Let $p \in [2, \infty)$. Prove the following interpolation inequality:

$$\forall u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \quad \|\nabla u\|_{L^p} \leq C\|u\|^{1/2}_{L^p}\|\nabla^2 u\|^{1/2}_{L^p}$$

where the constant C depands only on Ω and p.

6. Let μ be a (nonnegative) Radon measure on $\mathbf{R}^n$ and let $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a μ-measurable function, such that $|f|^p$ is also μ-integrable, for some $p \geq 1$. Prove that:

(i) $\displaystyle\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \ \mathrm{d}\mu = f(x) \qquad \mu - a.a.\ x,$

(ii) $\displaystyle\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)|^p \ \mathrm{d}\mu = 0 \qquad \mu - a.a.\ x.$

[Hint: Use differentiation of Radon measures.]

7. Let $\Omega \subset \mathbf{R}^n$ be open, bounded and of class $\mathcal{C}^1$. Let $u \in W^{1,p}(\Omega)$ for some $p \in [1, \infty)$. Show that Trace(u) is a nonpositive function (in the almost everywhere sense on $\partial\Omega$) iff the function $u^+ = \max(u, 0)$ belongs to $W^{1,p}_0(\Omega)$.