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Homework 25

1. (i) Prove that the space of all Borel measures on $\mathbf{R}^n$ with values in $\mathbf{R}^m$ is a linear space, and that it can be normed to a Banach space by setting: $\|\mu\| := |\mu|(\mathbf{R}^n)$.

(ii) Given an open set $U \subset \mathbf{R}^n$, let $[\mathcal{M}(U)]^m$ stand for the space of all Radon measures on $\mathbf{R}^n$, valued in $\mathbf{R}^m$, and concentrated on U, which means that: $|\mu|(\mathbf{R}^n \setminus U) = 0$. Prove that $[\mathcal{M}(U)]^m$ is a closed subset of the Banach space defined in (i), and therefore it is a Banach space itself.

2. Let μ be a positive, real measure on some measure space $(X, \mathcal{M})$. Let $f : X \longrightarrow \mathbf{R}^m$ be a Borel function, whose every component is integrable with respect to μ. Define:

$$\forall A \in \mathcal{M} \quad (\mu\lfloor f)(A) = \int_A f \, \mathrm{d}\mu.$$

Prove that $\mu\lfloor f$ is a $\mathbf{R}^m$ valued mesure and that its total variation is given by:

$$\|\mu\lfloor f\| = \int_X |f| \, \mathrm{d}\mu.$$

3. Let μ and ν be two real (signed) measures. Prove that:

(i) There is a real measure $\mu \wedge \nu$ which is smaller than μ and ν but is larger than any other measure which is smaller than μ and ν.

(ii) There is a real measure $\mu \vee \nu$ which is larger than μ and ν but is smaller than any other measure which is larger than μ and ν.

(iii) There holds: $\mu \wedge \nu + \mu \vee \nu = \mu + \nu$.

(iv) If μ and ν are nonnegative, then they are mutually singular iff $\mu \wedge \nu = 0$.

4. Let Ω be a bounded, open subset of $\mathbf{R}^n$. What is the dual of the Banach space of real continuous functions $\mathcal{C}(\bar{\Omega})$? (define and prove).

[Hint: Use the Riesz representation theorem]

In problems 5 and 6, μ is a real measure on some measure space $(X, \mathcal{M})$.

5. Prove that:

(i)

$$\forall A \in \mathcal{M} \qquad \mu^+(A) = \sup\{\mu(B);\ B \in \mathcal{M},\ B \subset A\},$$
$$\mu^-(A) = \sup\{-\mu(B);\ B \in \mathcal{M},\ B \subset A\}.$$

(ii) There exist disjoint sets $X^+, X^- \in \mathcal{M}$ such that: $\mu^+ = \mu\lfloor X^+$ and $\mu^- = \mu\lfloor X^-$. To do it, define X^+ to be the maximal positive set. That is $\mu(A) \geq 0$ for all $A \subset X^+$, $A \in \mathcal{M}$ and if $\tilde{X}^+$ has the same property then there must be $|\mu|(\tilde{X}^+ \setminus X^+) = 0$. Likewise, define X^- to be the maximal negative set.

6. In the context of problem 5, prove that one can have: $X = X^+ \cup X^-$ and $X^+ \cap X^- = \emptyset$. This is called the Hahn decomposition of X with respect to μ. Show that μ^+ and μ^- are mutually singular and that the decomposition $\mu = \mu^+ - \mu^-$ is unique.