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Homework 3

1. (i) Let $(E, \|\cdot\|)$ be the Banach space defined in problem 5 of Homework 1. For each $x \in X$, let $T_x(f) = f(x)$ Prove that each $T_x \in E^*$ and that $\|T_x - T_y\| = d(x,y)$. Deduce that X is therefore isometric to a subset of E^*.

(ii) Let E be a linear normed space. Prove that it is isometric with a linear subspace of the Banach space $\mathcal{B}(B_{E^*}(1))$ of bounded real functions on the closed unit ball in the dual space E^*: $B_{E^*}(1) = \{T \in E^*;\ \|T\| \leq 1\}$.

2. Prove that for every $x \neq y$ in a normed space E, there exists $T \in E^*$ such that $T(x) < T(y)$.

3. Let E be a Banach space and let F be its finitely dimensional subspace. Show that there exists a CLOSED subspace $G \subset E$ such that:

$$F + G = E \quad \text{and} \quad F \cap G = \{0\}.$$

[Hint: Use Hahn-Banach Theorem.]

4. In the linear space c_0 of all sequences of real numbers converging to 0, consider the sequence $\{e^i\}_{i=1}^{\infty}$, where $e^i = (0, 0, 0, \ldots 1, 0 \ldots)$ is such that 1 is only on the i-th place in the sequence e^i.

(i) Prove that $\{e^i\}_{i=1}^{\infty}$ is a Schauder basis in l_2.

(ii) Prove that $\{e^i\}_{i=1}^{\infty}$ is a Schauder basis in c_0. Recall that c_0 is normed by the l_∞ norm.

5. Prove that every vector space has a Hamel basis. [Hint: Use Zorn's lemma.]

6. Recall that a metric space Y is an extensor, if for every continuous function $f : A \longrightarrow Y$ defined on a closed subset A od a mteric space X, there exists a continuous function $\tilde{f} : X \longrightarrow Y$ such that $\tilde{f}(x) = f(x)$ for every $x \in A$.

Prove that:

(i) a space homeomorphic to an extensor is also an extensor,

(ii) a retract of an extensor is an extensor,

(iii) if Y_1 and Y_2 are extensors, then $Y_1 \times Y_2$ is an extensor.

7. Find the norms of the following linear functionals on $\mathcal{C}[-1,1]$:

(i) $T(f) := \int_0^1 f(x)\ \mathrm{d}x$,

(ii) $T(f) := \int_{-1}^1 (\mathrm{sgn}\ x) f(x)\ \mathrm{d}x$,

(iii) $T(f) := \int_{-1}^1 f(x)\ \mathrm{d}x - f(0)$,

(iv) $T(f) := \frac{f(\epsilon)+f(-\epsilon)-2f(0)}{\epsilon^2}$,

(v) $T(f) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} f(1/n)$