Marta Lewicka, Math 8601, Fall 2006

Homework 4

1. Let c_0 be space of all real sequences converging to 0 and let c be the space of all real convergent sequences. We view c_0 and c as subspaces of the Banach space l_∞.

(i) Deduce that both c_0 and c are Banach.
(ii) Prove that $(c_0)^*$ is linearly isometric to l_1.
(iii) Prove that c^* is also linearly isometric to l_1.
(iv) Prove that c_0 is NOT linearly isometric to c.

2. Prove that the space $\mathcal{C}_b(\mathbf{R}^n)$ of bounded continuous functions on $\mathbf{R}^n$ (with the supremum norm $\|\cdot\|_\infty$) is not separable.
[Hint: Modify the proof of nonseparability of l_∞.]

3. Let $f : U \longrightarrow \mathbf{R}^m$, where U is an open subset of $\mathbf{R}^n$. Prove that:

(i) if f is differentiable (in the sense of Frechet) at some $x_0 \in U$ then for each $v \in \mathbf{R}^n \setminus \{0\}$ there holds: $f'(x_0)(v) = Mv$, where M is the matrix of partial derivatives of f at x_0,
(ii) if all partial derivatives of f exist and are continuous in U, then f is differentiable (in the sense of Frechet) at each $x_0 \in U$.

4. Let (Y, d) be a complete metric space and let $f : B \longrightarrow Y$ be a contractive mapping with Lipschitz constant $\alpha < 1$. Here $B \subset Y$ is an open ball centered at some $y_0 \in Y$ and radius $r > 0$. Prove that if $d(f(y_0), y_0) < (1-\alpha)r$ then f has a fixed point.

5. Let (X, d) be a complete metric space and $F : X \longrightarrow X$ a map such that for some $n > 1$ the composition of the function f with itself n times: $f^{(n)} : X \longrightarrow X$ is a contraction.

(i) Does f have to be continuous?
(ii) Prove that f has the unique fixed point in X.