Marta Lewicka, Math 8601, Fall 2006

Homework 5

1. Let Λ be a metric space, E a Banach space, and let $F : \Lambda \times E \longrightarrow E$ be a function such that:

$$\exists \kappa < 1 \ \forall \lambda \in \Lambda \ \forall x, y \in E \quad \|F(\lambda, x) - F(\lambda, y)\| \leq \kappa \|x - y\|.$$

Prove that:

(i) for every $\lambda \in \Lambda$ there exists a unique $x(\lambda) \in E$ such that $x(\lambda) = F(\lambda, x(\lambda))$,

(ii) for every $\lambda \in \Lambda$, $y \in E$ one has:

$$\|x(\lambda) - F(\lambda, y)\| \leq \frac{\kappa}{1 - \kappa} \|y - F(\lambda, y)\|,$$

$$\|y - x(\lambda)\| \leq \frac{1}{1 - \kappa} \|y - F(\lambda, y)\|.$$

2. Show that a linear function $T : E \longrightarrow F$ between two Banach spaces E and F is differentiable (at any point) iff $T \in \mathcal{L}(E, F)$.

3. Let E be a Banach space. Show that the mapping $Inv : \mathcal{GL}(E, E) \longrightarrow \mathcal{GL}(E, E)$ given by $Inv(T) = T^{-1}$ is differentiable and find its derivative.

4. Let U be an open subset of a Banach space E. Given a function $g \in \mathcal{C}^1(U, \mathbf{R})$, define the mapping

$$S_g : \mathcal{C}([0, 1], U) \longrightarrow \mathbf{R}, \qquad S_g(f) = \int_0^1 g(f(s)) \, \mathrm{d}s.$$

Show that S_g is $\mathcal{C}^1$ and find its derivative.

5. Let $f_1 : E \longrightarrow F_1$ and $f_2 : E \longrightarrow F_2$ be two differentiable mappings between Banach spaces E, F_1, F_2.. Define $f : E \longrightarrow F_1 \times F_2$ by: $f(x) = (f_1(x), f_2(x))$. Prove that f is differentiable and find its derivative. (Here $F_1 \times F_2$ is a Banach space equipped with the norm $\|(x, y)\| := \|x\|_{F_1} + \|y\|_{F_2}$.)

6. Let E, F, G be normed spaces and let $\phi : E \times F \longrightarrow G$ be a bilinear map.

(i) Prove that ϕ is continuous iff it is bounded, that is:

$$\exists C > 0 \quad \forall x \in E \ \forall y \in F \qquad \|\phi(x, y)\| \leq C \cdot \|x\| \cdot \|y\|.$$

(ii) Let $\mathcal{L}(E, F; G)$ be the linear space of all continuous bilinear mappings ϕ, as above. Prove that it is a normed space, with the norm defined as:

$$\|\phi\| := \sup \{\|\phi(x, y)\|; \ \|x\| \leq 1, \ \|y\| \leq 1\|\} .$$

(iii) Prove that if G is Banach then $\mathcal{L}(E, F; G)$ is also Banach.