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Homework 6

1. Prove that a linear functional on a normed space is bounded iff its kernel is closed.

2. Let E be a Banach space and let E_0 be its closed subspace. Prove that the formula $\|x\|_1 = \inf\{\|y\|;\ y - x \in E_0\}$ defines a norm on the quotient space $E_1 = E/E_0$ and that $(E_1, \|\cdot\|_1)$ is a Banach space.

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{C}^1$ maps from an open subset U of a Banach space E into a Banach space F. Assume that $\{f_n\}$ converge pointwise to a map $f : U \longrightarrow F$ and that the sequence of derivatives $\{f'_n\}$ converges uniformly to a mapping $g : U \longrightarrow \mathcal{L}(E, F)$. Prove that f is $\mathcal{C}^1$ and that $f' = g$.

4. Let $f \in \mathcal{C}^k(U, F)$ where U is an open subset of a Banach space E and F is another Banach space. Let $x_0 \in U$ and $v \in E$ be such that $x_0 + tv \in U$, for every $t \in [0, 1]$. Prove the Taylor's formula:

$$f(x_0 + v) = f(x_0) + \left(\sum_{i=1}^{k} \frac{1}{i!} D^i f(x_0)(v, \dots v)\right) + R_k(x_0, v),$$

where $\|R_k(x_0, v)\|/\|v\|^k \to 0$ as $v \to 0$.

[Hint: As in the proof of the mean value theorem and the symmetricity of the second derivative, use the Hahn-Banach theorem to reduce everything to the case of a real function of one variable.]

5. Let $f \in \mathcal{C}^k(E, F)$ and $g \in \mathcal{C}^k(F, G)$, for some Banach spaces E, F, G and some natural $k \geq 1$. Prove that $g \circ f \in \mathcal{C}^k(E, G)$.

6. a) Let X be an infinite set and let M be the family of all finite subsets of X and their complements. Is M a σ-algebra?
b) Let X be an uncountable set and M the family of all countable subsets of X and their complements. Prove that M is a σ-algebra and that:

$$\mu(A) := \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise} \end{cases}$$

is a measure on M.

7. Let M be a σ-algebra of subsets of X, let N be a σ-algebra of subsets of Y. Given is a function $f : X \longrightarrow Y$. Which of the following families is a σ-algebra?
a) $\{B \subset Y;\ f^{-1}(B) \in M\}$,
b) $\{f(A);\ A \in M\}$,
c) $\{A \subset X;\ f(A) \in N\}$,
d) $\{f^{-1}(B);\ B \in N\}$.