Marta Lewicka, Math 8601, Fall 2006

Homework 7

1. Let μ be a measure on a σ-algebra $\mathcal{M}$ of subsets of X. Let $\mathcal{M}_1$ be a family of all sets $A \subset X$ of the form $A = B \cup C$, where $B \in \mathcal{M}$ and $C \subset D \in \mathcal{M}$, $\mu(D) = 0$. Show that $\mathcal{M}_1$ is a σ-algebra. Further, show that $\mu_1(A) := \mu(B)$ defines a complete measure μ_1 (this measure is called the completion of μ).

2. (i) Let μ^* be an external measure on X. Assume that $\mu^*(A) + \mu^*(X \setminus A) = \mu^*(X) < \infty$. Does A have to be measurable?
(ii) Give an example of an external measure which is not a measure.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $A \subset X$, define:

$$\mu^*(A) := \inf\{\mu(B);\ A \subset B,\ B \in \mathcal{M}\}.$$

(i) Show that μ^* is an external measure, coinciding with μ on $\mathcal{M}$ and such that it is 0 on every subset of a zero μ-measure set.
(ii) Let $\mathcal{M}_c$ be the σ-algebra generated by μ^*. Show that $\mathcal{M} \subset \mathcal{M}_c$.
(iii) Is the following characterisation true?:

$$\mathcal{M}_c = \left\{A \in 2^X;\ \exists B \in \mathcal{M}\ \mu(B) = \mu^*(A) \text{ and } \mu^*(B \setminus A) = 0\right\}$$

4. Let $f, \{f_n\}_{n=1}^\infty : X \longrightarrow \mathbf{R}$ be μ-measurable functions, for some measure μ on a σ-algebra $\mathcal{M} \subset 2^X$. We say that f_n *converges almost everywhere* to f if:

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for every point x outside some subset of measure 0.

We say that f_n *converges in measure* to f if for every $\epsilon > 0$:

$$\lim_{n\to\infty} \mu\left(x \in X;\ |f_n(x) - f(x)| \geq \epsilon\right) = 0.$$

(i) Prove that if μ is a finite measure then convergence almost everywhere implies convergence in measure.
(ii) In (i), can we omit the assumption of finiteness of μ?
(iii) In (i), is the converse implication true?

5. In the setting of problem 4:

(i) Prove that if f_n converges almost everywhere to some f and if f_n converges almost everywhere to another function g, then $f(x) = g(x)$ for every point x outside some subset of measure 0.
(ii) The same for convergence in measure.