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Homework 8

1. Let $f : X \longrightarrow \overline{\mathbf{R}}$ be measurable with respect to some σ-algebra $\mathcal{M}$ of subsets of X. Prove that for every Borel set $A \subset \mathbf{R}$ there holds: $f^{-1}(A) \in \mathcal{M}$.

2. Let $f : [a, b] \longrightarrow \mathbf{R}$ be a given function.

(i) If f is continuous, show that its graph is a set of (Lebesgue) measure 0 in $\mathbf{R}^2$.

(ii) What if f is just a (possibly discontinuous) monotone function?

3. Prove that the following subsets of $[0, 1]$ are compact, of Lebesgue measure 0 and uncountable:

(i) the set A containing all numbers which admit a binary representation $0, c_1c_2c_3...$ such that $c_n = 0$ for all n odd,

(ii) the set B of all numbers which admit a binary representation $0, c_1c_2c_3...$ such that for every n there is: $c_n = 0$ or $c_{n+1} = 0$.

4. Let $V \subset \mathbf{R}$ be a (nonmeasurable) Vitali's set.

(i) Prove that every (Lebesgue) measurable subset of V must have mesaure 0.

(ii) Prove that the following is a σ-algebra:

$$\mathcal{M} := \{A \triangle B;\ A \in \mathcal{L}_1,\ B \subset V\}$$

(here $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of sets A and B). Notice that $\mathcal{M}$ contains $\mathcal{L}_1$ and V.

(iii) Prove that the Lebesgue mesure can be extended to a complete measure on $\mathcal{M}$.

5. Prove that a set has Lebesgue measure 0 if and only if it can be covered infinitely many times by a sequence of boxes $\{B_n\}$ such that the series $\sum_n \text{vol}(B_n)$ converges. (The set is covered infinitely many times by a sequence of sets $\{B_n\}$ iff each point in this set belongs to infinitely many sets B_n.)

6. Recall that $\mathcal{B}(\mathbf{R}^n)$ denotes the σ-algebra of Borel subsets of $\mathbf{R}^n$. Let $A \subset \mathcal{B}(\mathbf{R}^n)$. For $k < n$ define:

$$A_k := \{(x_1, \ldots, x_k) \in \mathbf{R}^k;\ (x_1, \ldots, x_k, 0, \ldots, 0) \in A\}.$$

Prove that $A_k \in \mathcal{B}(\mathbf{R}^k)$.

7. Let A be a countable, infinite set. Prove that there exists an uncountable family $\{S_\alpha\}_{\alpha \in I}$ of subsets of A, such that:

(i) each set S_α is infinite,

(ii) if $\alpha \neq \beta$ then $S_\alpha \cap S_\beta$ is finite.

[Hint: Without loss of generality, we may think that $A = Q$ and $I = \mathbf{R} \setminus Q$.]