Marta Lewicka, Math 8601, Fall 2006

Homework 9

1. Let $f : [a, b] \longrightarrow \mathbf{R}$ be a given (Lebesgue) measurable function. Show that its graph is a set of (Lebesgue) measure 0 in $\mathbf{R}^2$.

2. Let $f : [a, b] \longrightarrow \mathbf{R}$ be a given function. Must its graph be a set of (Lebesgue) measure 0 in $\mathbf{R}^2$?

3. Let $A \subset C \subset \mathbf{R}^n$ and assume that $C \in \mathcal{L}_n$ and $\mu(C) < \infty$. Let μ^* denote the outer Lebesgue measure in $\mathbf{R}^n$. Prove that the following conditions are equivalent:

(i) $A \in \mathcal{L}_n$,
(ii) $\mu^*(A) + \mu^*(C \setminus A) = \mu^*(C)$.

4. Prove that every subset of $\mathbf{R}^n$ having positive outer Lebesgue measure contains a nonmeasurable subset.

5. Show that the derivative of a differentiable function $f : (a, b) \longrightarrow \mathbf{R}$ is a (Lebesgue) measurable function.

<u>Proof that c_0 in not complemented in l_∞:</u>

Consider the quotient space $M = l_\infty/c_0$ (see problem 2 Homework 6). If $x \in l_\infty$, then by $[x]$ we will denote the corresponding element of M.

Let $\{S_\alpha;\ \alpha \in \mathbf{R}\}$ be an uncountable family of subsets of $\mathbf{N}$, satisfying the properties (i) and (ii) in problem 7 Homework 8. For each $\alpha \in \mathbf{R}$, define $x_\alpha \in l_\infty$ in the following manner:

$$x_\alpha = \{x_\alpha^i\}_{i=1}^\infty; \qquad x_\alpha^i = \begin{cases} 1 & \text{if } i \in S_\alpha \\ 0 & \text{if } i \notin S_\alpha \end{cases}.$$

6. (a) Prove that if $\alpha \neq \beta$ then $[x_\alpha] \neq [x_\beta]$.

(b) Let $T \in M^*$. Prove that for every n-tuple $\alpha_1, \dots \alpha_n$ of distinct numbers in $\mathbf{R}$ there holds:

$$\left\| \left[\sum_{i=1}^n \text{sign } (T([x_{\alpha_i}]) \cdot x_{\alpha_i} \right] \right\|_M = 1.$$

(c) Deduce from (b) that for each $T \in M^*$ and each $n \in \mathbf{N}$ there are only finitely many $\alpha \in \mathbf{R}$ such that $|T([x_\alpha])| \geq 1/n$.

Therefore there are only countably many $\alpha \in \mathbf{R}$ so that $T([x_\alpha]) \neq 0$.

7. (a) Let N be any closed subset of l_∞. Prove that there exists a sequence $\{T_i\}_{i=1}^\infty$ of elements of N^* such that:

$$\forall x \neq y \in N \quad \exists i \qquad T_i(x) \neq T_i(y).$$

(b) From the above and from the conclusion of problem 6 deduce that c_0 is not complemented in l_∞.