MIDTERM 1, Math 8601, Fall 2006

Solve problem 1, and any two of the problems 2, 3 and 4. Clearly indicate your choice. Good luck!

Problem 1. (25 points: 5 points each, 1 point for correct answer, 4 points for correct explanation) Let E be a Banach space. True or False?

(i) For every $T \in E^*$ there exists a nonzero $x \in E$ such that $|T(x)| = ||T||\cdot||x||$.
False: $f \in \mathcal{C}[0,1] \mapsto T(f) = \int_0^{1/2} f - \int_{1/2}^1 f$.

(ii) For every $x \in E$ there exists a nonzero $T \in E^*$ such that $|T(x)| = ||T||\cdot||x||$.
True: by Hahn-Banach.

(iii) A linear function $T : E \longrightarrow E$ is continuous iff the preimage of the unit ball has a nonempty interior.
True: by definition of continuity.

(iv) Both the range and the kernel of any operator $T \in \mathcal{L}(E, E)$ are closed.
False: $f \in \mathcal{C}[0,1] \mapsto T(f)(\cdot) = \int_0^{\cdot} f \in \mathcal{C}[0,1]$.

(v) The linear space $R[x]$ of polynomials in the variable x with real coefficients can be normed to a Banach space.
False: it has a countable Hamel basis.

Problem 2. (25 points) Show that all norms on a linear space E are equivalent if and only of E is finitely dimensional.

Let $E = R^n$, we will prove that any norm $\|\cdot\|$ is equivalent to the Euclidean norm $\|\cdot\|_2$. Firstly, $\|x\| \leq \sum_{i=1}^n |x_i|\|e_i\| \leq C\sum|x_i| \leq C\sqrt{n}\|x\|_2$. In particular, the function $\|\cdot\|$ is Lipschitz continuous and nonzero on the sphere S^{n-1} in $(R^n, \|\cdot\|_2)$ and therefore it is bounded from below on this compact set by some $c > 0$. Hence: $\|x\| = \|x\|_2 \cdot \|x/\|x\|_2\| \geq \|x\|_2 \cdot c$, which ends the proof.

On the other hand, if E has (infinite) Hamel basis $\{e_\lambda\}_{\lambda\in\Lambda}$, then any family of positive real numbers $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ induces a norm on E given by: $\|x\| = \|\sum x_\lambda e_\lambda\| := \sum \alpha_\lambda |x_\lambda|$. Clearly, norms induced by a bounded and an unbounded family $\{\alpha_\lambda\}$ cannot be equivalent.

Problem 3. (25 points) Prove that for Banach spaces, the separability of the dual space E^* implies the separability of the base space E.

[Hint: From the dense sequence $T_n \in E^*$ derive the sequence of elements of E with unit norm and such that $T_n(x_n) \geq ||T_n||/2$.]

Following the hint, we prove that the subspace $F = span\ (\{x_n\})$ is dense in E. This will be enough, because the countable set of all rational linear combinations of finitely many elements of the sequence $\{x_n\}$ will be then dense in E. Now, if the closure of F is not the whole E, then by Hahn-Banach theorem there exists a functional $T \in E^*$, which is zero on F and for some point $x \in E \setminus F$ with unit norm we have: $T(x) = 1$. But on the other hand:

$$|T(x)| \leq |T_n(x) - T(x)| + |T_n(x)| \leq \epsilon + 2|T_n(x_n)| \leq \epsilon + 2|T_n(x_n) - T(x_n)| \leq 3\epsilon,$$

since for every $\epsilon > 0$ we may find T_n such that $||T_n - T|| \leq \epsilon$. Contradiction and end of proof.

Problem 4. (25 points) Let E be a linear normed space. Prove that E may be identified with a linear subspace of the space $\mathcal{B}(\bar{B}_{E^*})$, by means of a linear isometry. Here $\bar{B}_{E^*}$ stands for the closed unit ball in E^*. Must this linear subspace be closed?

Given $x \in E$, define $f_x : \bar{B}_{E^*} \longrightarrow R$ by: $f_x(T) := T(x)$. Clearly $x \mapsto f_x$ is linear and $\|f_x\|_\infty \leq \|x\|_E$. On the other hand, by Hanh-Banach theorem, for each x there is $T \in E^*$ with $\|T\| = 1$ and $T(x) = \|x\|$. Thus $\|f_x\|_\infty = \|x\|_E$.

Since the space $\mathcal{B}(\bar{B}_{E^*})$ is Banach, the linear space in question is closed iff it is Banach, which happens iff E is Banach, because an isometry preserves Cauchy sequences.